**MINISTRY OF EDUCATION AND TRAINING**
**THE UNIVERSITY OF DALAT**

**PHAM PHU PHAT**

# SOME PROPERTIES OF POLYNOMIAL MAPS IN TERMS OF NEWTON POLYHEDRONS

**Speciality: Mathematical Analysis**

**Speciality code: 9460102**

A THESIS

SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS
FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY IN MATHEMATICS

DA LAT, 2023

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**Supervisors:**

**1. Prof. Pham Tien Son**

**2. Dr. Dinh Si Tiep**

# Declaration of Authorship

I declare that this thesis titled, "SOME PROPERTIES OF POLYNOMIAL MAPS IN TERMS OF NEWTON POLYHEDRONS" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

# Introduction

Newton polyhedron has many applications in branches of mathematics such as Algebraic Geomatry, Geometry, Topology,... For instances, in Algebraic Geometry, Newton polyhedron is used as a tool to count the number of roots of a system of equations in $\mathbb{C}^n$ (Bernstein, 1975; Kouchnirenko, 1976; Li & Wang, 1996). In topology, (Kouchnirenko, 1976) computed the Milnor number of a complex polynomial satisfying the convenience and non-degeneracy condition in term of Newton polyhedron. The compactness of algebraic set and the global monodromy of a complex function are also the significance problems of topology which has also been conducted by others (Bodin, 2004; Phạm, 2008; Stalker, 2007). Łojasiewicz inequality is one of important topics in Geometry and Singularity theory which are paid attention by mathematicians. Hence, the computation and estimation of the Łojasiewicz exponent are interesting problems, especially, for a class of functions satisfying non-degenerate conditions in terms of Newton polyhedron (Bierstone & Milman, 1988; H. V. Hà & Phạm, 2014; Kuo, 1974).

The main purposes of this thesis are to study some properties of polynomial maps including monodromies and the compactness of algebraic sets defined by a class of polynomials satisfying non-degeneracy condition in terms of Newton polyhedron with some tools of Singularity theory and Semi-Algebraic Geometry.

More precisely, monodromies are the study of how objects from mathematic analysis, algebraic topology, ect., behave as they run round a singularity. The global monodromies of functions are defined by the following way.

Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial function. In the seventies (Thom, 1969), (Varchenko, 1972), (Verdier, 1996) and (Wallace, 1971) proved that there exists a finite set $B \subset \mathbb{C}$ named *the bifurcation set of* $f$, such that the restriction map

$$f\colon \mathbb{C}^n \setminus f^{-1}(B) \to \mathbb{C} \setminus B$$

is a locally trivial $C^\infty$-fibration. This fibration permits us to introduce the *global monodromy* of $f$. Namely, for $r > \max\{|c| \mid c \in B\}$ and $\mathbb{S}^1_r := \{c \in \mathbb{C} \mid |c| = r\}$, this is the restriction map

$$f\colon f^{-1}(\mathbb{S}^1_r) \to \mathbb{S}^1_r.$$

The problem of studying the bifurcation set and global monodromy of polynomial functions has been extensively studied in several papers. We would like to prefer the reader to (Artal-Bartolo, Luengo, & Melle-Hernéndez, 2000; Bodin, 2003, 2004; Broughton, 1988; Dimca & Némethi, 2001; Durfee, 1998; Hà, 1989, 1990, 1991; Hà & Lê, 1984; Hà & Nguyễn, 1989; Hà & Phạm, 1997; Hà & Zaharia, 1996; Kurdyka, Orro, & Simon, 2000; Némethi & Zaharia, 1990, 1992; Neumann & Norbury, 2000; Parusiński, 1995; Phạm, 2008, 2010; Rabier, 1997; Siersma & Tibăr, 1995, 1998; Tibăr, 1997), etc., and for the general case to (Hà & Nguyen, 2008; Jelonek, 2004; Jelonek & Kurdyka, 2005). However, we like to study in detail results of (Némethi & Zaharia, 1990) which are about the information of the bifurcation set in term of Newton polyhedron. For instance,

**Theorem 0.1.** *Let $f\colon \mathbb{C}^n \to \mathbb{C}$ and $S = \mathbb{C}^n$, we have*

$$B(f) \quad \subset \quad K_0(f) \cup T_\infty(f).$$

**Theorem 0.2.** *Assume that $f$ is not convenient, Newton non-degenerate at infinity and $f(0) = 0$. Then*

$$T_\infty(f) \quad \subset \quad \Sigma_\infty(f) \cup K_0(f) \cup \{0\}.$$

In the early eighties, in terms of Newton polyhedrons, M. Oka established the criterion for the stability of global monodromies for a family of polynomials satisfy the non-degeneracy condition (Oka, 1982). In details,

**Theorem 0.3.** *Suppose that $f$ and $g$ are analytic functions with the same Newton boundary and that they are Newton non-degenerate at infinity. Then their Milnor fibrations are isomorphic.*

In this thesis, the above theorems inspire us to study a global monodromy of a complex polynomial function restricting to a non-singular algebraic set $S \subset \mathbb{C}^n$. Its stability for the class of complex polynomial functions that satisfy the non-degeneracy condition is also investigated.

Another problem which attracts our studying is the compactness and the stable compactness of the real algebraic sets. More precisely, let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a nonconstant polynomial and $\mathcal{Z}(f)$ its zero set. We would like to know, firstly, when the set $\mathcal{Z}(f)$ is compact, and, secondly, when the set $\mathcal{Z}(f)$ is stably compact in the sense that it remains compact for all sufficiently small perturbations of the coefficients of the polynomial $f$.

In the univariate case, it is easy to see that $\mathcal{Z}(f)$ is a finite set, and is stably compact.

In the two-dimensional case (i.e., $n = 2$), (Stalker, 2007) provides a necessary criterion and a sufficient condition for the compactness of $\mathcal{Z}(f)$.

**Theorem 0.4** (Necessity)**.** *Assume that $\mathcal{Z}(f)$ is compact. Then*

*(1)* $f_{|Ox} \not\equiv 0 \not\equiv f_{|Oy}$.

*(2) One of following statements is true*

*(2.1)* $f$ *is bounded from below and* $f_\Delta(x, y) \geq 0, (x, y) \in \mathbb{R}^2, \Delta \in \Gamma_\infty(f)$.

*(2.2)* $f$ *is bounded from above and* $f_\Delta(x, y) \leq 0, (x, y) \in \mathbb{R}^2, \Delta \in \Gamma_\infty(f)$.

**Theorem 0.5** (Sufficiency). *Assume that*

*(1)* $f_{|Ox} \not\equiv 0 \not\equiv f_{|Oy}$.

*(2) One of following statements holds*

*(2.1)* $f_\Delta(x, y) > 0, (x, y) \in (\mathbb{R} \setminus \{0\})^2, \Delta \in \Gamma_\infty(f)$.

*(2.2)* $f_\Delta(x, y) < 0, (x, y) \in (\mathbb{R} \setminus \{0\})^2, \Delta \in \Gamma_\infty(f)$.

*Then* $\mathcal{Z}(f)$ *is compact.*

These conditions can be stated in terms of the Newton polyhedron of the polynomial $f$. However, his clever argument is not easy to extend to the higher dimension cases.

(Marshall, 2003, Theorem 5.1) gives a necessary and sufficient condition for the stable compactness of sets described by polynomial inequalities in terms of homogeneous components of highest degrees of the defining polynomials. In detail

**Theorem 0.6.** *Let* $K_S$ *be a the basic closed semi algebraic set defined by* $\{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, ..., s\}$ *and we denote* $v_i := \deg(g_i)$. *Then, we have*

*(1)* $K_S$ *is stably compact if and only if the function* $\max\{-g_{1v_1}, \ldots, -g_{sv_s}\}$ *is strictly positive on the unit sphere.*

*(2) If* $\epsilon > 0$ *is a lower bound for the function* $\max\{-g_{1v_1}, \ldots, -g_{sv_s}\}$ *on the unit sphere, then* $K_S$ *lies in the ball centered at the origin with radius*

$$r_\epsilon = \max\{1, \sum_{|\gamma| < v_i} |b_{i\gamma}| \big/ \epsilon : i = 1, \ldots, s\},$$

*where* $b_{i\gamma}$ *is the coefficient of* $x^\gamma$ *in* $g_i$ *and* $g_{ij}$ *is the homogeneous component of* $g_i$ *of degree* $j$.

Inspired by the above works, assuming that $n \geq 2$, we present two conditions for the compactness of $\mathcal{Z}(f)$, one necessary and one sufficient. From this we derive necessary and sufficient criteria for the stable compactness of $\mathcal{Z}(f)$. All these conditions are characterized in terms of the Newton polyhedron of the polynomial $f$.

This thesis is divided into three chapters.

Chapter 1 recalls some notions and results of Semi-algebraic Geometry, Newton polyhedron and the non-degeneracy condition that are useful for subsequent studies.

Chapter 2 (bases on the result [BP-2] in List of Author's Related Papers) investigates the bifurcation set and the monodromy of a complex polynomial function $f$ restricting to a non-singular algebraic set $S$ in terms of its Newton polyhedron where $f|_S$ is Newton non-degenerate at infinity. This fact implies that if $\{f_t\}_{t\in[0;1]}$ is a class of complex polynomials such that the Newton polyhedrons at infinity of $f_t|_S$ is independent of $t$ and the $f_t|_S$ is Newton non-degenerate at infinity, then the global monodromies of the $f_t|_S$ are all isomorphic. (see Theorem 2.5).

Chapter 3 (bases on the result [BP-1] in List of Author's Related Papers) establishes a necessary condition and a sufficient condition for the compactness of an algebraic set $\mathcal{Z}(f)$ which is defined by a real polynomial function which is bounded either from above or from below. This implies necessary and sufficient criteria for the stable compactness of $\mathcal{Z}(f)$. The mains results of this chapter are Theorem 3.1, Theorem 3.3 and Theorem 3.5.

# Contents

# Chapter 1

# Preliminaries

This chapter recalls some notions and results of Semi-algebraic Geometry and Newton polyhedron and the non-degeneracy condition. A detailed exposition, and proofs, can be found in (Dries, 1997; Gindikin, 1974; Hà & Phạm, 2017; J. Bochnak & Roy, 1998; Kouchnirenko, 1976; Lê, 2011; Mikhailov, 1967; Milnor, 1968; Némethi & Zaharia, 1992).

## 1.1 Semi-algebraic Geometry

This section begins with basic definitions of semi-algebraic sets and maps. Some notions and results of Semi-algebraic Geometry such as the Tarski–Seidenberg theorem, the Curve Selection Lemma... are also presented. A more detailed discussion and proofs can be found in (Dries, 1997; Hà & Phạm, 2017; J. Bochnak & Roy, 1998; Lê, 2011).

### 1.1.1 Semi-algebraic sets and maps

**Definition 1.1.** A subset of $\mathbb{K}^n$ is called *algebraic set* if it is of the form

$$\{x \in \mathbb{K}^n \mid f(x) = 0\},$$

where all $f$ are polynomials in $\mathbb{K}[x]$.

**Definition 1.2.** A subset of $\mathbb{R}^n$ is called *semi-algebraic* if it is a finite union of sets of the form

$$\{x \in \mathbb{R}^n \mid f_1(x) = 0; f_i(x) > 0, \quad i = 2, \ldots, k\},$$

where all $f_i$ are polynomials in $\mathbb{R}[x]$.

The following properties of semi-algebraic sets are elementary.

**Proposition 1.3.** *Let $A$ and $B$ be semi-algebraic subsets of $\mathbb{R}^n$. Then the sets $A \cup B, A \cap B$ and $\mathbb{R}^n \setminus A$ are also semi-algebraic.*

**Definition 1.4.** Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be semi-algebraic sets. A map $f\colon A \to B$ is said to be *semi-algebraic* if its graph

$$\mathrm{Graph}(f) = \{(x, y) \in A \times B \mid y = f(x)\}$$

is semi-algebraic in $\mathbb{R}^n \times \mathbb{R}^m$.

### 1.1.2 The Tarski–Seidenberg theorem

**Theorem 1.5.** (Tarski–Seidenberg theorem)(see (J. Bochnak & Roy, 1998)) *Let $A$ be a semi-algebraic subset of $\mathbb{R}^{n+m}$ and $\pi\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, the projection on the first $n$ coordinates. Then $\pi(A)$ is a semi-algebraic subset of $\mathbb{R}^n$.*

Let $x, y, z$ be variables ranging over the sets $X, Y, Z$, respectively, and let $\phi(x, y, z)$ and $\varphi(x, y, z)$ be *formulas* (conditions on $(x, y, z)$) defining the sets

$$\begin{aligned} \phi &:= \{(x, y, z) \in X \times Y \times Z \mid \phi(x, y, z) \text{ holds }\}, \\ \varphi &:= \{(x, y, z) \in X \times Y \times Z \mid \varphi(x, y, z) \text{ holds }\}. \end{aligned}$$

Then we can construct new formulas as below:

- The *disjunction* of $\phi$ and $\varphi$, denoted by $\phi \vee \varphi$, defines the set $\phi \cup \varphi$.
- The *conjunction* of $\phi$ and $\varphi$, denoted by $\phi \wedge \varphi$, defines the set $\phi \cap \varphi$.
- The *negation* of $\phi$, denoted by $\neg\phi$, defines the complement $X \times Y \times Z \setminus \phi$.
- The *existential quantification* over $z$ of $\phi(x, y, z)$, denoted by $\exists z \phi(x, y, z)$, defines the set $\{(x, y) \in X \times Y \mid \text{ there exists } z \in Z \text{ such that } \phi(x, y, z) \text{ holds } \}$.
- The *universal quantification* over $z$ of $\phi(x, y, z)$, denoted by $\forall z \phi(x, y, z)$, defines the set $\{(x, y) \in X \times Y \mid \text{ for all } z \in Z \text{ the condition } \phi(x, y, z) \text{ holds } \}$.

**Definition 1.6.** *A first-order formula* (of the language of ordered fields with parameters in $\mathbb{R}$) is obtained by the following rules.

(1) If $f \in \mathbb{R}[x_1, \ldots, x_n]$, then $f = 0$ and $f > 0$ are first-order formulas.

(2) If $\phi$ and $\varphi$ are first-order formulas, then $\phi \vee \varphi, \phi \wedge \varphi$ and $\neg\phi$ are also first-order formulas.

(3) If $\phi$ is a first-order formula and $x$ is a variable ranging over $\mathbb{R}$, then $\exists x \phi$ and $\forall x \phi$ are first-order formulas.

The formulas obtained by using only rules (1) and (2) are called *quantifier-free formulas.*

With the above notions, we have

**Theorem 1.7.** (Logical formulation of the Tarski–Seidenberg theorem)(see (Hà & Phạm, 2017)) *If $\phi(x)$ is a first-order formula, then the set $\{x \in \mathbb{R}^n \mid \phi(x) \text{ holds}\}$ is semi-algebraic.*

The following properties of semi-algebraic sets and maps follow from the Tarski–Seidenberg theorem.

**Proposition 1.8.** *The following statements hold.*

(i) *If $A$ and $B$ are semi-algebraic sets, then $A \times B$ is also semi-algebraic.*

(ii) *The closure, the interior and the boundary of a semi-algebraic set are semi-algebraic.*

(iii) *Images and inverse images of semi-algebraic sets under semi-algebraic maps are semi-algebraic.*

(iv) *Compositions of semi-algebraic maps are semi-algebraic.*

(v) *The sum and product of two semi-algebraic functions are semi-algebraic.*

### 1.1.3 Other results of Semi-algebraic Geometry

**Theorem 1.9.** (Curve Selection Lemma) (see (Hà & Phạm, 2017)) *Let $S$ be a semi-algebraic subset of $\mathbb{R}^n$ and $x_0 \in \bar{S} \setminus S$. Then there exists a real analytic semi-algebraic curve*

$$\phi\colon (0, \epsilon) \to S$$

*with $\phi(0) = x_0$ and with $\phi(t) \in S$ for $t \in (0, \epsilon)$.*

The following theorem plays an important role in proofs of our main results. For further details, we can see (Hà & Phạm, 2017).

**Theorem 1.10.** (Curve Selection Lemma at infinity) *Let $S \subset \mathbb{R}^n$ be a semi-algebraic set, and let*

$$f := (f_1, \ldots, f_p)\colon \mathbb{R}^n \to \mathbb{R}^p$$

*be a semi-algebraic map. Assume that there exists a sequence $\{x^k\}$ such that $x^k \in S, \lim_{k\to\infty} \|x^k\| = \infty$ and $\lim_{k\to\infty} f(x^k) = y \in (\overline{\mathbb{R}})^p$, where $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. Then there exists a smooth semi-algebraic curve*

$$\phi\colon (0, \epsilon) \to \mathbb{R}^n$$

*such that $\phi(t) \in S$ for all $t \in (0, \epsilon), \lim_{t\to 0} \|\phi(t)\| = \infty$ and $\lim_{t\to 0} f(\phi(t)) = y$.*

## 1.2 Newton polyhedron and non-degeneracy condition at infinity

This section covers topics: Newton polyhedron and the non-degeneracy conditions (at infinity) .

Throughout this section, we consider a fixed coordinate system $(x_1, \ldots, x_n) \in \mathbb{K}^n$ where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. We denote by $\mathbb{Z}_+$ the set of non-negative integer numbers. If $x = (x_1, \ldots, x_n) \in \mathbb{K}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, we denote the monomial $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ by $x^\alpha$ and by $|\alpha|$ the sum $\alpha_1 + \cdots + \alpha_n$.

For $J \subset \{1, \ldots, n\}$, let $\mathbb{K}^J := \{(x_1, \ldots, x_n) \mid x_i = 0 \text{ for } x_j \notin J\}$.

In case $\mathbb{K} = \mathbb{C}$, the gradient of a polynomial function $f \colon \mathbb{C}^n \to \mathbb{C}$ is denoted by $\nabla f$ as usual, i.e.,

$$\nabla f(x) := \left( \overline{\frac{\partial f}{\partial x_1}(x)}, \ldots, \overline{\frac{\partial f}{\partial x_n}(x)} \right),$$

so the differential of $f$ along $\mathbf{v}$ may be expressed by $\partial f / \partial \mathbf{v} = \langle \mathbf{v}, \nabla f \rangle$.

Let $f \colon \mathbb{K}^n \to \mathbb{K}$ be a polynomial function. Suppose that $f$ is written as $f = \sum_\alpha a_\alpha x^\alpha$. Then, the norm of $f$ is defined to be $|f| := \max_\alpha |a_\alpha|$. The support of $f$, denoted by $\mathrm{supp}(f)$, is defined as the set of those $\alpha \in \mathbb{Z}_+^n$ such that $a_\alpha \neq 0$. The *Newton polyhedron of $f$ (at infinity)* , denoted by $\Gamma(f)$, is defined as the convex hull in $\mathbb{R}^n$ of the set $\mathrm{supp}(f) \cup \{0\}$. The polynomial $f$ is said to be *convenient* if $\Gamma(f)$ intersects each coordinate axis in a point different from the origin 0 in $\mathbb{R}^n$. For each (closed) face $\Delta$ of $\Gamma(f)$, we will denote by $f_\Delta$ the polynomial $\sum_{\alpha \in \Delta} a_\alpha x^\alpha$; if $\Delta \cap \mathrm{supp}(f) = \emptyset$ we let $f_\Delta := 0$.

Given a nonzero vector $q := (q_1, \ldots, q_n) \in \mathbb{R}^n$, we define

$$\begin{aligned} d(q, \Gamma(f)) &:= \min \left\{ \sum_{i=1}^n q_i \alpha_i \mid \alpha \in \Gamma(f) \right\}, \\ \Delta(q, \Gamma(f)) &:= \left\{ \alpha \in \Gamma(f) \mid \sum_{i=1}^n q_i \alpha_i = d(q, \Gamma(f)) \right\}. \end{aligned}$$

By definition, for each nonzero vector $q \in \mathbb{R}^n$, $\Delta(q, \Gamma(f))$ is a closed face of $\Gamma(f)$. Conversely, if $\Delta$ is a closed face of $\Gamma(f)$ then there exists a nonzero vector[1] $q \in \mathbb{R}^n$ such that $\Delta = \Delta(q, \Gamma(f))$.

**Notation 1.11.** In chapter 3, we consider Newton polyhedron of $f$ (at infinity) as the convex hull in $\mathbb{R}^n$ of the set $\mathrm{supp}(f)$ and denote $\Gamma_\infty(f)$ by the Newton boundary of $f$ (at infinity) defined as the union of all faces $\Delta(q, \Gamma(f))$ for some $q \in \mathbb{R}^n$ with $\min_{j=1,\ldots,n} q_j < 0$.

**Remark 1.12.** The following statements follow immediately from definitions:

(i) For each nonempty subset $I$ of $\{1, \ldots, n\}$, if the restriction of $f$ on $\mathbb{C}^I$ is not identically zero, then $\Gamma(f) \cap \mathbb{R}^I = \Gamma(f|_{\mathbb{C}^I})$.

(ii) Let $\Delta := \Delta(q, \Gamma(f))$ for some nonzero vector $q := (q_1, \ldots, q_n) \in \mathbb{R}^n$ and put $d := d(q, \Gamma(f))$. By definition, $f_\Delta = \sum_{\alpha \in \Delta} a_\alpha x^\alpha$ is a weighted homogeneous polynomial of type $(q, d)$, i.e., we have for all $t > 0$ and all $x \in \mathbb{C}^n$,

$$f_\Delta(t^{q_1} x_1, \ldots, t^{q_n} x_n) = t^d f_\Delta(x_1, \ldots, x_n).$$

This implies the Euler relation

$$\sum_{i=1}^{n} q_i x_i \frac{\partial f_\Delta}{\partial x_i}(x) = d f_\Delta(x).$$

In particular, if $d \neq 0$ and $\nabla f_\Delta(x) = 0$, then $f_\Delta(x) = 0$.

Let $g_1, \ldots, g_p \colon \mathbb{C}^n \to \mathbb{C}$ be polynomial functions.

**Definition 1.13.** *The algebraic set $S := \{x \in \mathbb{C}^n \mid g_1(x) = 0, \ldots, g_p(x) = 0\}$ is non-singular if the system of of gradient vectors*

$$\nabla g_1(x), \ldots, \nabla g_p(x)$$

*is $\mathbb{C}$-linearly independent for all $x \in S$.*

[1]Since $\Gamma(f)$ is an integer polyhedron, we can assume that all the coordinates of $q$ are rational numbers.

The following definition of the non-degeneracy condition is inspired from the work of Kouchnirenko (Kouchnirenko, 1976), where the case $S = \mathbb{C}^n$ was considered.

**Definition 1.14.** We say that the restriction of $f$ on $S$ is *Newton non-degenerate (at infinity)* if, and only if, for every nonempty subset $I \subset \{1, \ldots, n\}$ with $f|_{\mathbb{C}^I} \not\equiv 0$, for every (possibly empty) subset $J \subset \{j \in \{1, \ldots, p\} \mid g_j|_{\mathbb{C}^I} \not\equiv 0\}$, and for every vector $q \in \mathbb{R}^n$ with $\min_{i \in I} q_i < 0$, the following conditions hold:

(i) the set

$$\{x \in \mathbb{C}^{*n} \mid g_{j,\Delta_j}(x) = 0 \text{ for } j \in J\}$$

is a reduced smooth complete intersection variety in the torus $\mathbb{C}^{*n}$, i.e., the system of gradient vectors $\nabla g_{j,\Delta_j}(x)$ for $j \in J$ is $\mathbb{C}$-linearly independent on this variety;

(ii) if $d(q, \Gamma(f|_{\mathbb{C}^I})) < 0$, then the set

$$\{x \in \mathbb{C}^{*n} \mid f_{\Delta_0}(x) = 0 \text{ and } g_{j,\Delta_j}(x) = 0 \text{ for } j \in J\}$$

is a reduced smooth complete intersection variety in the torus $\mathbb{C}^{*n}$;

where $\Delta_0 := \Delta(q, \Gamma(f|_{\mathbb{C}^I}))$ and $\Delta_j := \Delta(q, \Gamma(g_j|_{\mathbb{C}^I}))$ for $j \in J$.

**Notation 1.15.** In chapter 3, we consider the above condition in the case of the variables ranging over $\mathbb{R}$ and $S = \mathbb{R}^n$.

Finally, following (Némethi & Zaharia, 1990), we introduce a set, which plays an important role in Chapter 2.

**Definition 1.16.** We denote $\Sigma_\infty(f|_S)$ by the set of all values $c \in \mathbb{C}$ for which there exist a nonempty set $I \subset \{1, \ldots, n\}$ with $f|_{\mathbb{C}^I} \not\equiv 0$, a (possibly empty) set $J \subset \{j \in \{1, \ldots, p\} : g_j|_{\mathbb{C}^I} \not\equiv 0\}$, a vector $q \in \mathbb{R}^n$ with $\min_{i \in I} q_i < 0$ and $d(q, \Gamma(f|_{\mathbb{C}^I})) = 0$, a point $x \in \mathbb{C}^{*I}$, and scalars $\lambda_j \in \mathbb{C}$ for $j \in J$, such that the following conditions hold:

$$
\begin{aligned}
c &= f_{\Delta_0}(x), \\
g_{j,\Delta_j}(x) &= 0 \text{ for } j \in J, \\
\nabla f_{\Delta_0}(x) + \sum_{j \in J} \lambda_j \nabla g_{j,\Delta_j}(x) &= 0,
\end{aligned}
$$

where $\Delta_0 := \Delta(q, \Gamma(f|_{\mathbb{C}^I}))$ and $\Delta_j := \Delta(q, \Gamma(g_j|_{\mathbb{C}^I}))$ for $j \in J$.

We observe that the above value $c \in \Sigma_\infty(f|_S)$ is indeed a critical value of the restriction of the polynomial $f_{\Delta_0}$ on the variety

$$\{x \in \mathbb{C}^{*I} \ : \ g_{j,\Delta_j}(x) = 0 \quad \text{for} \quad j \in J\}.$$

Hence, by the Bertini–Sard theorem, $\Sigma_\infty(f|_S)$ is a finite set provided that the restriction $f|_S$ is Newton non-degenerate at infinity.

## 1.3 Bertini-Sard theorem

The below knowledge of this section can be found in (Spivak, 1965).

**Definition 1.17.** A set $M$ of $\mathbb{R}^n$ is called a $k-$ dimensional *manifold* if for every point $x \in M$, there is an open set $U$ containing $x, V \subset \mathbb{R}^k$ and diffeomorphism $\varphi \colon U \to V$ such that $\varphi(U \cap M) = V$. We call $(\varphi, U)$ a chart of $M$.

**Theorem 1.18.** Let M $\subset \mathbb{R}^n$ and $\Theta$ be an open cover of $M$. Then, there exist a collection $\Phi$ of $C^\infty$ functions $\varphi$ defined in open sets containing M with the following properties:

(i) For each $x \in M$, we have $0 \le \varphi(x) \le 1$.

(ii) For each $x \in M$ there is an open set $V$ containing $x$ such that all but finitely many $\varphi \in \Phi$ are 0 on $V$.

(iii) For each $x \in M$, we have $\sum_{\varphi \in \Phi} \varphi(x) = 1$.

(iv) For each $\varphi \in \Phi$, there is an open set $U$ in $\Theta$ such that $\varphi = 0$ outside of some closed set contain in $U$.

A collection $\Phi$ is called *a $C^\infty$ partition of unity* of M.

**Definition 1.19.** Let $M, N$ be submanifolds of a manifold $Y$ . $M$ transversally with $N$ if at every point $x \in M \cap N$, we have $T_xM + T_xN = T_xY$.

Let $f \colon A \to B$ be a smooth map of a manifold $A$ to $B$.

**Definition 1.20.** The map $f$ is called a locally trivial fibration if for any $b \in B$ there exists a neighbourhood $U$ of $B$ and a diffeomorphism (called the trivialization) from $f^{-1}(U)$ to $U \times f^{-1}(b)$ preserving the projection to $U$.

**Theorem 1.21.** (Bertini-Sard theorem) *The measure of the set of critical values of $f$ is equal to zero.*

# Chapter 2

# Bifurcation Sets and Global Monodromies of Newton Non-degenerate Polynomials on Algebraic Sets

Let $S \subset \mathbb{C}^n$ be a non-singular algebraic set and $f\colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial function. It is well-known that the restriction $f|_S\colon S \to \mathbb{C}$ of $f$ on $S$ is a locally trivial fibration outside a finite set $B(f|_S) \subset \mathbb{C}$. In this chapter, we give an explicit description of a finite set $T_\infty(f|_S) \subset \mathbb{C}$ such that $B(f|_S) \subset K_0(f|_S) \cup T_\infty(f|_S)$, where $K_0(f|_S)$ denotes the set of critical values of the $f|_S$. Furthermore, $T_\infty(f|_S)$ is contained in the set of critical values of certain polynomial functions provided that the $f|_S$ is Newton non-degenerate at infinity. Using these facts, we show that if $\{f_t\}_{t\in[0,1]}$ is a family of polynomials such that the Newton polyhedron at infinity of $f_t$ is independent of $t$ and the $f_t|_S$ is Newton non-degenerate at infinity, then the global monodromies of the $f_t|_S$ are all isomorphic.

## 2.1 The bifurcation set of a polynomial function

From now on, let $g_1, \ldots, g_p \colon \mathbb{C}^n \to \mathbb{C}$ be polynomial functions such that the algebraic set

$$S := \{x \in \mathbb{C}^n \mid g_1(x) = 0, \ldots, g_p(x) = 0\}$$

is non-singular.

Let $f \colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial function. It is well known that the bifurcation set $B(f|_S)$ of the restriction $f|_S \colon S \to \mathbb{C}$ contains the set $K_0(f|_S)$. Recall that we write $K_0(f|_S)$ for *the set of critical values* of the restriction of $f$ on $S$, i.e.,

$$K_0(f|_S) := \{c \in \mathbb{C} \quad | \quad \exists x \in S, \exists \lambda_j \in \mathbb{C}, j = 1, \ldots, p, \text{ such that } f(x) = c \quad \text{and} \quad \nabla f(x) + \sum_{j=1}^{p} \lambda_j \nabla g_j(x) = 0\}.$$

By theorem Logical formulation of the Tarski-Seidenberg theorem (see Theorem 1.7), $K_0(f|_S)$ is a semi-algebraic set in $\mathbb{C}$ and hence, by the Bertini–Sard theorem, $K_0(f|_S)$ is a finite set.

Before formulating our first theorem, we also need the following concept (see also (Hà & Phạm, 2017, Chapter 2)).

**Definition 2.1.** By the *set of tangency values at infinity* of $f|_S$ we mean the set

$$T_\infty(f|_S) := \{c \in \mathbb{C} \mid \exists \{x^{(k)}\} \subset S, \exists \{\lambda_j^{(k)}\} \subset \mathbb{C}, j = 1, \ldots, p+1, \|x^{(k)}\| \to \infty, f(x^{(k)}) \to c, \ \nabla f(x^{(k)}) + \sum_{j=1}^{p} \lambda_j^{(k)} \nabla g_j(x^{(k)}) = \lambda_{p+1}^{(k)} x^{(k)} \text{ for all } k \in \mathbb{N}\}.$$

The results of Theorem 0.1 and Theorem 0.2 inspire us to introduce two following theorems.

**Theorem 2.2.** *$T_\infty(f|_S)$ is a finite set and the following inclusion holds*

$$B(f|_S) \subset K_0(f|_S) \cup T_\infty(f|_S). \tag{2.1}$$

Under the non-degeneracy condition of Definition 1.14, we obtain the following bound of tangency values at infinity of $f|_S$ in terms of critical values of certain polynomial functions.

**Theorem 2.3.** *Assume that the restriction $f|_S$ of $f$ on $S$ is Newton non-degenerate at infinity. Then*

$$T_\infty(f|_S) \quad \subset \quad \Sigma_\infty(f|_S) \cup K_0(f|_S) \cup \{0\}.$$

*Moreover, if the polynomial $f\colon \mathbb{C}^n \to \mathbb{C}$ is convenient, then $T_\infty(f|_S) = \emptyset$.*

For $S = \mathbb{C}^n$, the next statement was shown in (Némethi & Zaharia, 1990, Theorem 2).

**Corollary 2.4.** *Under the assumption of Theorem 2.3, we have*

$$B(f|_S) \quad \subset \quad \Sigma_\infty(f|_S) \cup K_0(f|_S) \cup \{0\}.$$

*Moreover, if the polynomial $f\colon \mathbb{C}^n \to \mathbb{C}$ is convenient, then $B(f|_S) = K_0(f|_S)$.*

## 2.2 The stability of global monodromies

By the inspiration of Theorem 0.3 , we begin our works about the stability result. Recall that the (non-singular) algebraic set $S$ is given by

$$S := \{x \in \mathbb{C}^n \mid g_1(x) = 0, \ldots, g_p(x) = 0\}.$$

In what follows, let $f\colon \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}, (t, x) \mapsto f(t, x)$, be a polynomial function. We will write $f_t(x) := f(t, x)$ and assume that for each $t \in [0, 1]$, the restriction $f_t|_S\colon S \to \mathbb{C}$ is dominant (i.e., the image set $f_t(S)$ is dense in $\mathbb{C}$).

**Theorem 2.5.** *Let the following conditions are satisfied:*

*(i) The Newton polyhedron of $f_t$ is independent of $t$;*

*(ii) For each $t \in [0, 1]$, the restriction $f_t|_S$ is Newton non-degenerate at infinity.*

*Then the global monodromies of the $f_t|_S$ are all isomorphic.*

# Chapter 3

# Compactness criteria for real algebraic set and Newton polyhedron

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial and $\mathcal{Z}(f)$ its zero set. In this chapter, in terms of the so-called Newton polyhedron of $f$, we present a necessary criterion and a sufficient condition for the compactness of $\mathcal{Z}(f)$. From this we derive necessary and sufficient criteria for the stable compactness of $\mathcal{Z}(f)$.

## 3.1 The compactness of an algebraic set.

From now on let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a nonconstant polynomial in $n \geq 2$ variables and let $\mathcal{Z}(f)$ be its zero set:

$$\mathcal{Z}(f) := \{x \in \mathbb{R}^n \mid f(x) = 0\}.$$

The results of Theorem 0.4 and 0.5 are the roots of the following works which focus on the case $n \geq 2$.

**Theorem 3.1.** *Suppose that $\mathcal{Z}(f)$ is compact. The following assertions hold true:*

*(i)* $f|_{\mathbb{R}^J} \not\equiv 0$ *for all* $J \subset \{1, \ldots, n\}$,

*(ii) One of the following statements holds*

*(ii1)* $f$ *is bounded from below and* $f_\Delta \geq 0$ *on* $\mathbb{R}^n$ *for all* $\Delta \in \Gamma_\infty(f)$.

*(ii2)* $f$ *is bounded from above and* $f_\Delta \leq 0$ *on* $\mathbb{R}^n$ *for all* $\Delta \in \Gamma_\infty(f)$.

The following remark shows that the converse of Theorem 3.1 does not hold.

**Remark 3.2.** Let $n = 2$ and consider the polynomial

$$f(x_1, x_2) := (x_1 - x_2)^2.$$

By definition, the Newton polyhedron $\Gamma(f)$ is a segment joining the two points $(2, 0)$ and $(0, 2)$, and so the Newton boundary $\Gamma_\infty(f)$ is the union of the faces:

$$\Delta_1 := \{(2, 0)\}, \quad \Delta_2 := \{(0, 2)\}, \quad \text{and} \quad \Delta_3 := \{(1 - t)(2, 0) + t(0, 2) \mid 0 \leq t \leq 1\}.$$

Clearly, the polynomials $f_{\Delta_1}(x_1, x_2) = x_1^2, f_{\Delta_2}(x_1, x_2) = x_2^2$, and $f_{\Delta_3}(x_1, x_2) = (x_1 - x_2)^2$ are all non-negative on $\mathbb{R}^n$. However, $\mathcal{Z}(f) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = x_2\}$ is not compact. On the other hand, we have the following statement, which provides a sufficient condition for compactness of real algebraic sets.

**Theorem 3.3.** *Suppose the following conditions:*

*(i)* $f|_{\mathbb{R}^J} \not\equiv 0$ *for all* $J \subset \{1, \ldots, n\}$,

*(ii) One of the following statements holds*

*(ii1)* $f_\Delta > 0$ *on* $(\mathbb{R} \setminus \{0\})^n$ *for all* $\Delta \in \Gamma_\infty(f)$.

*(ii2)* $f_\Delta < 0$ *on* $(\mathbb{R} \setminus \{0\})^n$ *for all* $\Delta \in \Gamma_\infty(f)$.

*Then* $\mathcal{Z}(f)$ *is compact.*

## 3.2 The stability of compactness of an algebraic set.

In the rest of this chapter we study the stable compactness of real algebraic sets, which is easier to check than compactness.

**Definition 3.4.** The set $\mathcal{Z}(f)$ is called *stably compact* if there is $\epsilon > 0$ such that $\mathcal{Z}(f+g)$ is compact for all polynomials $g\colon \mathbb{R}^n \to \mathbb{R}$ with $\Gamma(g) \subseteq \Gamma(f)$ and $|g| < \epsilon$.

By definition, the set $\mathcal{Z}(f)$ is stably compact if, and only if, remains compact for all sufficiently small perturbations of the "Newton" coefficients of the polynomial $f$.

**Theorem 3.5** (Compare Theorem 0.6)**.** *The following conditions are equivalent:*

*(i)* $\mathcal{Z}(f)$ *is stably compact.*

*(ii)* $f|_{\mathbb{R}^J} \not\equiv 0$ *for all* $J \subset \{1, \ldots, n\}$ *and* $f_\Delta \neq 0$ *on* $(\mathbb{R} \setminus \{0\})^n$ *for all* $\Delta \in \Gamma_\infty(f)$.

*(iii)* $f|_{\mathbb{R}^J} \not\equiv 0$ *for all* $J \subset \{1, \ldots, n\}$ *and one of the following statements holds*

*(iii1)* $f_\Delta > 0$ *on* $(\mathbb{R} \setminus \{0\})^n$ *for all* $\Delta \in \Gamma_\infty(f)$.

*(iii2)* $f_\Delta < 0$ *on* $(\mathbb{R} \setminus \{0\})^n$ *for all* $\Delta \in \Gamma_\infty(f)$.

*(iv)* $f|_{\mathbb{R}^J} \not\equiv 0$ *for all* $J \subset \{1, \ldots, n\}$ *and there exist* $\sigma \in \{-1, 1\}$ *and constants* $c_1 > 0, c_2 > 0,$ *and* $R > 0$ *such that*

$$c_1\mathcal{P}(x) \ \leq \ \sigma f(x) \ \leq \ c_2\mathcal{P}(x) \quad \textit{for all} \quad \|x\| > R. \tag{3.1}$$

*(v)* $f|_{\mathbb{R}^J} \not\equiv 0$ *for all* $J \subset \{1, \ldots, n\}$, $f$ *is Newton non-degenerate at infinity, and there exist* $\sigma \in \{-1, 1\}$ *and* $R > 0$ *such that* $\sigma f(x) \geq 0$ *for all* $\|x\| > R$.

# Conclusions

The main goals of this thesis are to study properties of a class of functions satisfying non-degenerate conditions. Singularity Theory and Semi-algebraic Geometry are main tools for our study. Our main results include:

- Investigating the global monodromy of a family polynomial $\{f_t\}$ restricting on an algebraic set in which the Newton polyhedrons of $\{f_t\}$ are independent from $t$ and satisfy the non-degenerated condition. (see Theorem 2.5).

- Giving a necessary condition and a sufficient condition for the compactness of an algebraic set $\mathcal{Z}(f)$ which is defined by a real polynomial function which is bounded either from above or from below. This implies the necessary and sufficient criteria for the stable compactness of $\mathcal{Z}(f)$. (see Theorem 3.1, Theorem 3.3 and Theorem 3.5).

# List of Author's Related Papers

[BP-1] P. P. Phạm and T. S. Phạm, *Compactness criteria for real algebraic set and Newton polyhedra,* Forum Mathematicum, **30** (6)(2018).

[BP-2] T. T. Nguyen, P. P. Phạm and T. S. Phạm, *Bifurcation Sets and Global Monodromies of Newton Non-degenerate Polynomials on Algebraic Sets,* PRIMS Kyoto Univ., **55** (4) (2019).

# References


Artal-Bartolo, E., Luengo, I., & Melle-Hernéndez, A. (2000). Milnor number at infinity, topology and Newton boundary of a polynomial function. *Math. Z.*, *233*(4), 679–696.

Bernstein, D. N. (1975). The number of roots of a system of equations. *Funktsional. Anal. Prilozhen.*, *3*(9), 1–4.

Bierstone, E., & Milman, P. D. (1988). Semianalytic and subanalytic sets. *Publ. Math.I.H.E.S., Bures-sur-Yvette, France*(67), 5–42.

Bodin, A. (2003). Invariance of Milnor numbers and topology of complex polynomials. *Comment. Math. Helv.*, *78*(1), 134–152.

Bodin, A. (2004). Newton polygons and families of polynomials. *manuscripta math.*, *113*(3), 371–382.

Broughton. (1988). Milnor numbers and the topology of polynomial hypersurfaces. *Invent. Math.*, *92*, 217–242.

Dimca, A., & Némethi, A. (2001). On the monodromy of complex polynomials. *Duke Math. Journal*, *108*(2), 199–209.

Dries, L. D. (1997). Tame topology and o-minimal structures. *LMS Lecture Notes, Cambridge University Press*, 1031–1035.

Durfee, A. (1998). Five definitions of critical points at infinity. In *Singularities* (Vol. 162, pp. 345–360). Basel: Birkhäuser.

Gindikin, S. G. (1974). Energy estimates connected with the Newton polyhedron. *(English) Trans. Moscow Math. Soc.*, *31*, 189–236.

Hà, H. V. (1989). Sur la fibration globale des polynômes de deux variables complexes. *C. R. Acad. Sci., Série I. Math.*, *309*, 231–234.

Hà, H. V. (1990). Nombres de Łojasiewicz et singularités à l'infini des polynômes de deux variables complexes. *C. R. Acad. Sci., Paris, Série I*, *311*, 429–432.

Hà, H. V. (1991). Sur l'irrégularité du diagramme splice pour l'entrelacement à l'infini des courbes planes. *C. R. Acad. Sci., Paris, Série I*, *313*(5), 277–280.

Hà, H. V., & Lê, D. T. (1984). Sur la topologie des polynômes complexes. *Acta Math. Vietnam.*, *9*, 21–32.

Hà, H. V., & Nguyễn, L. A. (1989). Le comportement géométrique à l'infini des polynômes de deux variables complexes. *C. R. Acad. Sci., Paris, Série I*, *309*(3), 183–186.

Hà, H. V., & Nguyen, T. T. (2008). On the topology of polynomial functions on algebraic surfaces in $\mathbb{C}^n$. In J. P. Brasselet, J. L. Cisneros-Molina, D. Massey, J. Seade, & B. Teissier (Eds.), *Singularities ii: Geometric and topological aspects* (Vol. 475, pp. 61–67). Providence, RI: Amer. Math. Soc.

Hà, H. V., & Phạm, T. S. (1997). Invariance of the global monodromies in families of polynomials of two complex variables. *Acta Math. Vietnam.*, *22*(2), 515–526.

Hà, H. V., & Phạm, T. S. (2017). *Genericity in polynomial optimization* (Vol. 3). Singapore: World Scientific Publishing.

Hà, H. V., & Zaharia, A. (1996). Families of polynomials with total Milnor number constant. *Math. Ann.*, *313*, 481–488.

H. V. Hà, N. T. T., S. T. Dinh, & Phạm, T. S. (2014). Global łojasiewicz-type inequality for nondegenerate polynomial maps. *J. Math. Anal. Appl.*, *2*(410), 541–560.

J. Bochnak, M. C., & Roy, M.-F. (1998). Real algebraic geometry. *Springer*, *36*(1), 59–79.

Jelonek, Z. (2004). On asymptotic critical values and the Rabier theorem. *Banach Center Publ.*, *65*, 125–133.

Jelonek, Z., & Kurdyka, K. (2005). Quantitative generalized Bertini–Sard theorem for smooth affine varieties. *Discrete Comput. Geom.*, *34*(4), 659–678.

Kouchnirenko, A. G. (1976). Polyhedres de Newton et nombre de Milnor. *Invent. Math.*, *32*, 1–31.

Kuo, T. C. (1974). Computation of lojasiewicz exponent of $f(x, y)$. *Comment. Math. Helv.*(49), 201–213.

Kurdyka, K., Orro, P., & Simon, S. (2000). Semialgebraic Sard theorem for generalized critical values. *J. Differential Geom.*, *56*, 62–92.

Lê, L. T. (2011). Nhập môn hình học giai tích thục. *Đại học Đà Lạt*.

Li, T. Y., & Wang, X. (1996). The bbk root count in $\mathbb{C}^n$. *Math. Comp.*, *216*(65).

Marshall, M. (2003). Optimization of polynomial functions. *Canad. Math. Bull.*, *46*(4), 575–587.

Mikhailov, V. P. (1967). The behaviour at infinity of a class of polynomials. *(Russian) Trudy Mat. lnst. Steklov, (English) Proc. Steklov Inst. Math., Vol*, *91*(1), 61–82.

Milnor, J. (1968). *Singular points of complex hypersurfaces* (Vol. 61). Princeton: Princeton University Press.

Némethi, A., & Zaharia, A. (1990). On the bifurcation set of a polynomial function and Newton boundary. *Publ. Res. Inst. Math. Sci.*, *26*(4), 681–689.

Némethi, A., & Zaharia, A. (1992). Milnor fibration at infinity. *Indag. Math.*, *3*, 323–335.

Neumann, W. D., & Norbury, P. (2000). Monodromy and vanishing cycles of complex polynomials. *Duke Math. Journal*, *101*(4), 487–497.

Oka, M. (1982). On the topology of the Newton boundary iii. *J. Math. Soc. Japan*, *34*(3), 541–549.

Parusiński, A. (1995). On the bifurcation set of a complex polynomial with isolated singularities at infinity. *Compos. Math.*, *97*, 369–384.

Phạm, T. S. (2008). On the topology of the Newton boundary at infinity. *J. Math. Soc. Japan*, *60*(4), 1065–1081.

Phạm, T. S. (2010). Invariance of the global monodromies in families of nondegenerate polynomials in two variables. *Kodai Math. J.*, *33*(2), 294–309.

Rabier, P. J. (1997). Ehresmann fibrations and Palais–Smale conditions for morphisms of Finsler manifolds. *Ann. of Math.*, *146*, 647–691.

Siersma, D., & Tibăr, M. (1995). Singularities at infinity and their vanishing cycles. *Duke Math. J.*, *80*(3), 771–783.

Siersma, D., & Tibăr, M. (1998). Topology of polynomial functions and monodromy dynamics. *C. R. Acad. Sci. Paris Sér. I Math.*, *327*(9), 655–660.

Spivak, M. (1965). *Calculus on manifolds.* the United States of America: Addison-Wesley Publishing.

Stalker, J. (2007). A compactness criterion for real plane algebraic curves. *Forum Math*, *19*, 563–570.

Thom, R. (1969). Ensembles et morphismes stratifiés. *Bull. Amer. Math. Soc.*, *75*, 240–284.

Tibăr, M. (1997). On the monodromy fibration of polynomial functions with singularities at infinity. *C. R. Acad. Sci. Paris Sér. I Math.*, *9*(1), 1031–1035.

Varchenko, A. N. (1972). Theorems on the topological equisingularity of families of algebraic varieties and families of polynomial mappings. *Math. USSR Izv.*, *6*, 949–1008.

Verdier, J. L. (1996). Stratifications de Whitney et theorème de Bertini–Sard. *Invent. Math.*, *36*, 295–312.

Wallace, A. H. (1971). Linear sections of algebraic varieties. *Indiana Univ. Math. J.*, *20*, 1153–1162.