MINISTRY OF EDUCATION AND TRAINING
THE UNIVERSITY OF DALAT

PHAM PHU PHAT

SOME PROPERTIES OF POLYNOMIAL MAPS IN TERMS OF NEWTON POLYHEDRONS

Speciality: Mathematical Analysis

Speciality code: 9460102

A THESIS

SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS
FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY IN MATHEMATICS

DA LAT, 2023

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Supervisors:

1. Prof. Pham Tien Son

2. Dr. Dinh Si Tiep

Declaration of Authorship

I, Pham Phu Phat , declare that this thesis titled, "SOME PROPERTIES OF POLYNOMIAL MAPS IN TERMS OF NEWTON POLYHEDRONS" and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
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- I have acknowledged all main sources of help.
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Abstract

SOME PROPERTIES OF POLYNOMIAL MAPS IN TERMS OF NEWTON POLYHEDRONS

The goals of this thesis are to study properties of a class of functions satisfying non-degeneracy conditions. Singularity Theory and Semi-algebraic Geometry are main tools for our study.

Our main results include:

- Investigating the global monodromy of a class of complex polynomial $\{f_t\}$ restricting to a non-singular algebraic set in which the Newton polyhedrons of $\{f_t\}$ are independent of t and satisfy the non-degeneracy condition at infinity.

- Giving a necessary condition and a sufficient condition for the compactness of a algebraic set $\mathcal{Z}(f) := \{x \in \mathbb{R}^n \mid f(x) = 0\}$ in terms of the Newton polyhedron of f in the case where f is bounded either from above or from below. This implies the necessary and sufficient criteria for the stable compactness of $\mathcal{Z}(f)$.

Keywords and phrases: non-degeneracy conditions at infinity, Newton polyhedron.

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Introduction

Newton polyhedron has many applications in branches of mathematics such as Algebraic Geomatry, Geometry, Topology,... For instances, in Algebraic Geometry, Newton polyhedron is used as a tool to count the number of roots of a system of equations in $\mathbb{C}^n$ (Bernstein, 1975; Kouchnirenko, 1976; Li & Wang, 1996). In topology, (Kouchnirenko, 1976) computed the Milnor number of a complex polynomial satisfying the convenience and non-degeneracy condition in term of Newton polyhedron. The compactness of algebraic set and the global monodromy of a complex function are also the significance problems of topology which has also been conducted by others (Bodin, 2004; Phạm, 2008; Stalker, 2007). Łojasiewicz inequality is one of important topics in Geometry and Singularity theory which are paid attention by mathematicians. Hence, the computation and estimation of the Łojasiewicz exponent are interesting problems, especially, for a class of functions satisfying non-degenerate conditions in terms of Newton polyhedron (Bierstone & Milman, 1988; H. V. Hà & Phạm, 2014; Kuo, 1974).

The main purposes of this thesis are to study some properties of polynomial maps including monodromies and the compactness of algebraic sets defined by a class of polynomials satisfying non-degeneracy condition in terms of Newton polyhedron with some tools of Singularity theory and Semi-Algebraic Geometry.

More precisely, monodromies are the study of how objects from mathematic analysis, algebraic topology, ect., behave as they run round a singularity. The global monodromies of functions are defined by the following way.

Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial function. In the seventies (Thom, 1969), (Varchenko, 1972), (Verdier, 1996) and (Wallace, 1971) proved that there exists a finite set $B \subset \mathbb{C}$ named *the bifurcation set of* f, such that the restriction map

$$f\colon \mathbb{C}^n \setminus f^{-1}(B) \to \mathbb{C} \setminus B$$

is a locally trivial C^∞-fibration. This fibration permits us to introduce the *global monodromy* of f. Namely, for $r > \max\{|c| \mid c \in B\}$ and $\mathbb{S}^1_r := \{c \in \mathbb{C} \mid |c| = r\}$, this is the restriction map

$$f\colon f^{-1}(\mathbb{S}^1_r) \to \mathbb{S}^1_r.$$

The problem of studying the bifurcation set and global monodromy of polynomial functions has been extensively studied in several papers. We would like to prefer the reader to (Artal-Bartolo, Luengo, & Melle-Hernéndez, 2000; Bodin, 2003, 2004; Broughton, 1988; Dimca & Némethi, 2001; Durfee, 1998; Hà, 1989, 1990, 1991; Hà & Lê, 1984; Hà & Nguyễn, 1989; Hà & Phạm, 1997; Hà & Zaharia, 1996; Kurdyka, Orro, & Simon, 2000; Némethi & Zaharia, 1990, 1992; Neumann & Norbury, 2000; Parusiński, 1995; Phạm, 2008, 2010; Rabier, 1997; Siersma & Tibăr, 1995, 1998; Tibăr, 1997), etc., and for the general case to (Hà & Nguyen, 2008; Jelonek, 2004; Jelonek & Kurdyka, 2005). However, we like to study in detail results of (Némethi & Zaharia, 1990) which are about the information of the bifurcation set in term of Newton polyhedron. For instance,

Theorem 0.1. *Let $f\colon \mathbb{C}^n \to \mathbb{C}$ and $S = \mathbb{C}^n$, we have*

$$B(f) \quad \subset \quad K_0(f) \cup T_\infty(f).$$

Theorem 0.2. *Assume that f is not convenient, Newton non-degenerate at infinity and $f(0) = 0$. Then*

$$T_\infty(f) \quad \subset \quad \Sigma_\infty(f) \cup K_0(f) \cup \{0\}.$$

In the early eighties, in terms of Newton polyhedrons, M. Oka established the criterion for the stability of global monodromies for a family of polynomials satisfy the non-degeneracy condition (Oka, 1982). In details,

Theorem 0.3. *Suppose that f and g are analytic functions with the same Newton boundary and that they are Newton non-degenerate at infinity. Then their Milnor fibrations are isomorphic.*

In this thesis, the above theorems inspire us to study a global monodromy of a complex polynomial function restricting to a non-singular algebraic set $S \subset \mathbb{C}^n$. Its stability for the class of complex polynomial functions that satisfy the non-degeneracy condition is also investigated.

Another problem which attracts our studying is the compactness and the stable compactness of the real algebraic sets. More precisely, let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a nonconstant polynomial and $\mathcal{Z}(f)$ its zero set. We would like to know, firstly, when the set $\mathcal{Z}(f)$ is compact, and, secondly, when the set $\mathcal{Z}(f)$ is stably compact in the sense that it remains compact for all sufficiently small perturbations of the coefficients of the polynomial f.

In the univariate case, it is easy to see that $\mathcal{Z}(f)$ is a finite set, and is stably compact.

In the two-dimensional case (i.e., $n = 2$), (Stalker, 2007) provides a necessary criterion and a sufficient condition for the compactness of $\mathcal{Z}(f)$.

Theorem 0.4 (Necessity)**.** *Assume that $\mathcal{Z}(f)$ is compact. Then*

(1) $f_{|Ox} \not\equiv 0 \not\equiv f_{|Oy}$.

(2) One of following statements is true

(2.1) f *is bounded from below and* $f_\Delta(x, y) \geq 0, (x, y) \in \mathbb{R}^2, \Delta \in \Gamma_\infty(f)$.

(2.2) f *is bounded from above and* $f_\Delta(x, y) \leq 0, (x, y) \in \mathbb{R}^2, \Delta \in \Gamma_\infty(f)$.

Theorem 0.5 (Sufficiency). *Assume that*

(1) $f_{|Ox} \not\equiv 0 \not\equiv f_{|Oy}$.

(2) One of following statements holds

(2.1) $f_\Delta(x, y) > 0, (x, y) \in (\mathbb{R} \setminus \{0\})^2, \Delta \in \Gamma_\infty(f)$.

(2.2) $f_\Delta(x, y) < 0, (x, y) \in (\mathbb{R} \setminus \{0\})^2, \Delta \in \Gamma_\infty(f)$.

Then $\mathcal{Z}(f)$ *is compact.*

These conditions can be stated in terms of the Newton polyhedron of the polynomial f. However, his clever argument is not easy to extend to the higher dimension cases.

(Marshall, 2003, Theorem 5.1) gives a necessary and sufficient condition for the stable compactness of sets described by polynomial inequalities in terms of homogeneous components of highest degrees of the defining polynomials. In detail

Theorem 0.6. *Let* K_S *be a the basic closed semi algebraic set defined by* $\{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, ..., s\}$ *and we denote* $v_i := \deg(g_i)$*. Then, we have*

(1) K_S *is stably compact if and only if the function* $\max\{-g_{1v_1}, \ldots, -g_{sv_s}\}$ *is strictly positive on the unit sphere.*

(2) If $\epsilon > 0$ *is a lower bound for the function* $\max\{-g_{1v_1}, \ldots, -g_{sv_s}\}$ *on the unit sphere, then* K_S *lies in the ball centered at the origin with radius*

$$r_\epsilon = \max\{1, \sum_{|\gamma|<v_i} |b_{i\gamma}|/\epsilon : i = 1, \ldots, s\},$$

where $b_{i\gamma}$ *is the coefficient of* x^γ *in* g_i *and* g_{ij} *is the homogeneous component of* g_i *of degree* j.

Inspired by the above works, assuming that $n \geq 2$, we present two conditions for the compactness of $\mathcal{Z}(f)$, one necessary and one sufficient. From this we derive necessary and sufficient criteria for the stable compactness of $\mathcal{Z}(f)$. All these conditions are characterized in terms of the Newton polyhedron of the polynomial f.

This thesis is divided into three chapters.

Chapter 1 recalls some notions and results of Semi-algebraic Geometry, Newton polyhedron and the non-degeneracy condition that are useful for subsequent studies.

Chapter 2 (bases on the result [BP-2] in List of Author's Related Papers) investigates the bifurcation set and the monodromy of a complex polynomial function f restricting to a non-singular algebraic set S in terms of its Newton polyhedron where $f|_S$ is Newton non-degenerate at infinity. This fact implies that if $\{f_t\}_{t\in[0;1]}$ is a class of complex polynomials such that the Newton polyhedrons at infinity of $f_t|_S$ is independent of t and the $f_t|_S$ is Newton non-degenerate at infinity, then the global monodromies of the $f_t|_S$ are all isomorphic. (see Theorem 2.7).

Chapter 3 (bases on the result [BP-1] in List of Author's Related Papers) establishes a necessary condition and a sufficient condition for the compactness of an algebraic set $\mathcal{Z}(f)$ which is defined by a real polynomial function which is bounded either from above or from below. This implies necessary and sufficient criteria for the stable compactness of $\mathcal{Z}(f)$. The mains results of this chapter are Theorem 3.3, Theorem 3.5 and Theorem 3.10.

Table of Notations

$\mathbb{Z}$	set of integer numbers
$\mathbb{Z}_+$	set of non-negative integer numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}_+$	set of non-negative real numbers
$\mathbb{C}$	set of complex numbers
$\langle x, y\rangle$	canonical inner product
$\lvert x\rvert$	modulus, absolute value of x
$\lVert x\rVert$	Euclidean norm of a vector x
$\Gamma(f)$	Newton polyhedron of f at infinity
$\Gamma_\infty(f)$	Newton boundary of f at infinity
$B(f\rvert_S)$	bifurcation set of $f\rvert_S$
$K_0(f\rvert_S)$	set of critical values of $f\rvert_S$
$K_\infty(f\rvert_S)$	set of asymptotic critical values at infinity of $f\rvert_S$
$T_\infty(f\rvert_S)$	set of tangency values at infinity of $f\rvert_S$

Contents

Chapter 1

Preliminaries

This chapter recalls some notions and results of Semi-algebraic Geometry and Newton polyhedron and the non-degeneracy condition. A detailed exposition, and proofs, can be found in (Dries, 1997; Gindikin, 1974; Hà & Phạm, 2017; J. Bochnak & Roy, 1998; Kouchnirenko, 1976; Lê, 2011; Mikhailov, 1967; Milnor, 1968; Némethi & Zaharia, 1992).

1.1 Semi-algebraic Geometry

This section begins with basic definitions of semi-algebraic sets and maps. Some notions and results of Semi-algebraic Geometry such as the Tarski–Seidenberg theorem, the Curve Selection Lemma... are also presented. A more detailed discussion and proofs can be found in (Dries, 1997; Hà & Phạm, 2017; J. Bochnak & Roy, 1998; Lê, 2011).

1.1.1 Semi-algebraic sets and maps

Definition 1.1. A subset of $\mathbb{K}^n$ is called *algebraic set* if it is of the form

$$\{x \in \mathbb{K}^n \mid f(x) = 0\},$$

where all f are polynomials in $\mathbb{K}[x]$.

Definition 1.2. A subset of $\mathbb{R}^n$ is called *semi-algebraic* if it is a finite union of sets of the form

$$\{x \in \mathbb{R}^n \mid f_1(x) = 0; f_i(x) > 0, \quad i = 2, \ldots, k\},$$

where all f_i are polynomials in $\mathbb{R}[x]$.

Example 1.3. (i) The semi-algebraic subsets of $\mathbb{R}$ are the unions of finitely many points and open intervals.

(ii) Any algebraic subsets of $\mathbb{R}^n$ are semi-algebraic.

(iii) Let $f(b, c, x) = x^2 + bx + c$ be a polynomial. The set

$$\{(b, c) \in \mathbb{R}^2 \mid f \text{ has exactly 2 distinct real roots}\}$$

is semi-algebraic in $\mathbb{R}^2$.

(iv) The following sets are not semi-algebraic

$$\{(x, y) \in \mathbb{R}^2 \mid y = \sin x\}, \quad \{(x, y) \in \mathbb{R}^2 \mid y = e^x\}, \qquad \{(x, y) \in \mathbb{R}^2 \mid y = nx, n \in \mathbb{N}\}.$$

The following properties of semi-algebraic sets are elementary.

Proposition 1.4. *Let A and B be semi-algebraic subsets of $\mathbb{R}^n$. Then the sets $A \cup B, A \cap B$ and $\mathbb{R}^n \setminus A$ are also semi-algebraic.*

Definition 1.5. Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be semi-algebraic sets. A map $f\colon A \to B$ is said to be *semi-algebraic* if its graph

$$\mathrm{Graph}(f) = \{(x, y) \in A \times B \mid y = f(x)\}$$

is semi-algebraic in $\mathbb{R}^n \times \mathbb{R}^m$.

1.1.2 The Tarski–Seidenberg theorem

Theorem 1.6. (Tarski–Seidenberg theorem)(see (J. Bochnak & Roy, 1998)) *Let A be a semi-algebraic subset of $\mathbb{R}^{n+m}$ and $\pi\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, the projection on the first n coordinates. Then $\pi(A)$ is a semi-algebraic subset of $\mathbb{R}^n$.*

Let x, y, z be variables ranging over the sets X, Y, Z, respectively, and let $\phi(x, y, z)$ and $\varphi(x, y, z)$ be *formulas* (conditions on (x, y, z)) defining the sets

$$\begin{aligned} \phi &:= \{(x, y, z) \in X \times Y \times Z \mid \phi(x, y, z) \text{ holds } \}, \\ \varphi &:= \{(x, y, z) \in X \times Y \times Z \mid \varphi(x, y, z) \text{ holds } \}. \end{aligned}$$

Then we can construct new formulas as below:

- The *disjunction* of ϕ and φ, denoted by $\phi \vee \varphi$, defines the set $\phi \cup \varphi$.
- The *conjunction* of ϕ and φ, denoted by $\phi \wedge \varphi$, defines the set $\phi \cap \varphi$.
- The *negation* of ϕ, denoted by $\neg\phi$, defines the complement $X \times Y \times Z \setminus \phi$.
- The *existential quantification* over z of $\phi(x, y, z)$, denoted by $\exists z \phi(x, y, z)$, defines the set $\{(x, y) \in X \times Y \mid \text{ there exists } z \in Z \text{ such that } \phi(x, y, z) \text{ holds } \}$.
- The *universal quantification* over z of $\phi(x, y, z)$, denoted by $\forall z \phi(x, y, z)$, defines the set $\{(x, y) \in X \times Y \mid \text{ for all } z \in Z \text{ the condition } \phi(x, y, z) \text{ holds } \}$.

Definition 1.7. *A first-order formula* (of the language of ordered fields with parameters in $\mathbb{R}$) is obtained by the following rules.

(1) If $f \in \mathbb{R}[x_1, \dots, x_n]$, then $f = 0$ and $f > 0$ are first-order formulas.

(2) If ϕ and φ are first-order formulas, then $\phi \vee \varphi, \phi \wedge \varphi$ and $\neg\phi$ are also first-order formulas.

(3) If ϕ is a first-order formula and x is a variable ranging over $\mathbb{R}$, then $\exists x\phi$ and $\forall x\phi$ are first-order formulas.

The formulas obtained by using only rules (1) and (2) are called *quantifier-free formulas.*

With the above notions, we have

Theorem 1.8. (Logical formulation of the Tarski–Seidenberg theorem)(see (Hà & Phạm, 2017)) *If $\phi(x)$ is a first-order formula, then the set $\{x \in \mathbb{R}^n \mid \phi(x) \text{ holds}\}$ is semi-algebraic.*

Example 1.9. Let $f(b, c, x) = x^2 + bx + c$ be a real polynomial. The set

$$\exists x(x^2 + bx + c = 0) \ \wedge \ \exists y(y^2 + by + c = 0) \wedge \neg(x = y)$$

is a semi-algebraic and in fact it is

$$\{(b, c) \in \mathbb{R}^2 \mid b^2 - 4c > 0\}.$$

Example 1.10. Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial. The set

$$\begin{aligned} T_\infty(f) \ &:= \ \{c \in \mathbb{C} \mid \exists\{x^{(k)}\} \subset \mathbb{C}^n, \exists\{\lambda^{(k)}\} \subset \mathbb{C}, \|x^{(k)}\| \to \infty, \\ &\qquad f(x^{(k)}) \to c, \ \nabla f(x^{(k)}) = \lambda^{(k)} x^{(k)} \ \text{ for all } \ k \in \mathbb{N}\} \end{aligned}$$

described by the following formula

$$\forall\epsilon\exists x\exists\lambda((||x|| > \frac{1}{\epsilon}) \ \wedge \ (|f(x) - c| < \epsilon) \ \wedge \ (\nabla f(x) = \lambda x)).$$

Hence, $T_\infty(f)$ is a semi-algebraic set.

The following properties of semi-algebraic sets and maps follow from the Tarski–Seidenberg theorem.

Proposition 1.11. *The following statements hold.*

(i) *If A and B are semi-algebraic sets, then $A \times B$ is also semi-algebraic.*

(ii) *The closure, the interior and the boundary of a semi-algebraic set are semi-algebraic.*

(iii) *Images and inverse images of semi-algebraic sets under semi-algebraic maps are semi-algebraic.*

(iv) *Compositions of semi-algebraic maps are semi-algebraic.*

(v) *The sum and product of two semi-algebraic functions are semi-algebraic.*

Example 1.12. (i) Let $A \subset \mathbb{R}^n$ be a semi-algebraic set. If $F\colon A \to \mathbb{R}^m$ is a polynomial mapping, it is semi-algebraic.

(ii) Let $A \subset \mathbb{R}^n, A \neq \emptyset$ be a semi-algebraic set. Then the distance function

$$d(\cdot, A)\colon \mathbb{R}^n \to \mathbb{R}, \qquad x \mapsto d(x, A) := \inf\{\|x - a\| \mid a \in A\}$$

is continuous semi-algebraic.

1.1.3 Other results of Semi-algebraic Geometry

Theorem 1.13. (Curve Selection Lemma) (see (Hà & Phạm, 2017)) *Let S be a semi-algebraic subset of $\mathbb{R}^n$ and $x_0 \in \bar{S} \setminus S$. Then there exists a real analytic semi-algebraic curve*

$$\phi\colon (0, \epsilon) \to S$$

with $\phi(0) = x_0$ and with $\phi(t) \in S$ for $t \in (0, \epsilon)$.

The following theorem plays an important role in proofs of our main results. For further details, we can see (Hà & Phạm, 2017).

Theorem 1.14. (Curve Selection Lemma at infinity) *Let $S \subset \mathbb{R}^n$ be a semi-algebraic set, and let*

$$f := (f_1, \ldots, f_p)\colon \mathbb{R}^n \to \mathbb{R}^p$$

be a semi-algebraic map. Assume that there exists a sequence $\{x^k\}$ *such that* $x^k \in S, \lim_{k\to\infty} \|x^k\| = \infty$ *and* $\lim_{k\to\infty} f(x^k) = y \in (\overline{\mathbb{R}})^p$*, where* $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$*. Then there exists a smooth semi-algebraic curve*

$$\phi\colon (0,\epsilon) \to \mathbb{R}^n$$

such that $\phi(t) \in S$ *for all* $t \in (0,\epsilon), \lim_{t\to 0} \|\phi(t)\| = \infty$ *and* $\lim_{t\to 0} f(\phi(t)) = y$*.*

Proof. By replacing, if necessary, f_i by $\frac{\pm 1}{1 \pm (f_i(x))^2}$, we may assume that $y \in \mathbb{R}^p$. Consider the semi-algebraic map $\Phi : \mathbb{R}^n \mapsto \mathbb{R}^{n+1} \times \mathbb{R}^p$ given by

$$\Phi(x) := \left(\frac{x_1}{1+||x||^2}, ..., \frac{x_n}{1+||x||^2}, \frac{1}{1+||x||^2}, f(x) \right).$$

Without loss of generality, we can suppose that the sequence $\Phi(x_k)$ is convergent to some point $(u, y) \in \mathbb{S}^n \times \mathbb{R}^p$. By the Tarski–Seidenberg theorem, $\Phi(S)$ is a semi-algebraic set. Then by the Curve Selection Lemma, there exists an analytic semi algebraic curve

$$\psi\colon (-\epsilon,\epsilon) \to \mathbb{R}^n \times \mathbb{R}^p,$$

$$t \mapsto (\psi_1(t), ..., \psi_n(t), \psi_{n+1}(t), ..., \psi_{n+1+p}(t)),$$

such that $\psi(0) = (u, y)$ and $\psi(t) \in \Phi(S)$ for all $t \in (0,\epsilon)$. Now, define the curve $\phi : (0,\epsilon) \to \mathbb{R}^n, t \mapsto \phi(t)$, by

$$\phi(t) := \left(\frac{\psi_1(t)}{\psi_{n+1}(t)}, ..., \frac{\psi_n(t)}{\psi_{n+1}(t)} \right).$$

Then it is clear that ϕ has the required properties.

□

1.2 Newton polyhedron and non-degeneracy condition at infinity

This section covers topics: Newton polyhedron and the non-degeneracy conditions (at infinity) .

Throughout this section, we consider a fixed coordinate system $(x_1, \ldots, x_n) \in \mathbb{K}^n$ where $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. We denote by $\mathbb{Z}_+$ the set of non-negative integer numbers. If $x = (x_1, \ldots, x_n) \in \mathbb{K}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, we denote the monomial $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ by x^α and by $|\alpha|$ the sum $\alpha_1 + \cdots + \alpha_n$.

For $J \subset \{1, \ldots, n\}$, let $\mathbb{K}^J := \{(x_1, \ldots, x_n) \mid x_i = 0 \text{ for } x_j \notin J\}$.

In case $\mathbb{K} = \mathbb{C}$, the gradient of a polynomial function $f\colon \mathbb{C}^n \to \mathbb{C}$ is denoted by ∇f as usual, i.e.,

$$\nabla f(x) := \left(\overline{\frac{\partial f}{\partial x_1}(x)}, \ldots, \overline{\frac{\partial f}{\partial x_n}(x)} \right),$$

so the differential of f along $\mathbf{v}$ may be expressed by $\partial f / \partial \mathbf{v} = \langle \mathbf{v}, \nabla f \rangle$.

Let $\lambda\colon (0, \epsilon) \to \mathbb{C}$ be a power series with fractional exponents and assume that

$$\lambda(s) := a s^\alpha + \text{ (higher-order terms)},$$

where $a \neq 0$ and $\alpha \in \mathbb{Q}$. Then we can introduce the *order* of λ,

$$\mathrm{ord}_s \lambda(s) \ := \ \alpha.$$

Let $f\colon \mathbb{K}^n \to \mathbb{K}$ be a polynomial function. Suppose that f is written as $f = \sum_\alpha a_\alpha x^\alpha$. Then, the norm of f is defined to be $|f| := \max_\alpha |a_\alpha|$. The support of f, denoted by $\mathrm{supp}(f)$, is defined as the set of those $\alpha \in \mathbb{Z}_+^n$ such that $a_\alpha \neq 0$. The *Newton polyhedron of f (at infinity)* , denoted by $\Gamma(f)$, is defined as the convex hull in $\mathbb{R}^n$ of the set $\mathrm{supp}(f) \cup \{0\}$. The polynomial f is said to be *convenient* if $\Gamma(f)$ intersects each coordinate axis in a point different from the origin 0 in $\mathbb{R}^n$. For each (closed) face Δ of $\Gamma(f)$, we will denote by f_Δ the polynomial $\sum_{\alpha \in \Delta} a_\alpha x^\alpha$; if $\Delta \cap \mathrm{supp}(f) = \emptyset$ we let $f_\Delta := 0$.

Given a nonzero vector $q := (q_1, \ldots, q_n) \in \mathbb{R}^n$, we define

$$\begin{aligned} d(q, \Gamma(f)) \ &:= \ \min \left\{ \sum_{i=1}^n q_i \alpha_i \mid \alpha \in \Gamma(f) \right\}, \\ \Delta(q, \Gamma(f)) \ &:= \ \left\{ \alpha \in \Gamma(f) \mid \sum_{i=1}^n q_i \alpha_i = d(q, \Gamma(f)) \right\}. \end{aligned}$$

By definition, for each nonzero vector $q \in \mathbb{R}^n$, $\Delta(q, \Gamma(f))$ is a closed face of $\Gamma(f)$. Conversely, if Δ is a closed face of $\Gamma(f)$ then there exists a nonzero vector[1] $q \in \mathbb{R}^n$ such that $\Delta = \Delta(q, \Gamma(f))$.

Notation 1.15. In chapter 3, we consider Newton polyhedron of f (at infinity) as the convex hull in $\mathbb{R}^n$ of the set $\mathrm{supp}(f)$ and denote $\Gamma_\infty(f)$ by the Newton boundary of f (at infinity) defined as the union of all faces $\Delta(q, \Gamma(f))$ for some $q \in \mathbb{R}^n$ with $\min_{j=1,\ldots,n} q_j < 0$.

Remark 1.16. The following statements follow immediately from definitions:

(i) For each nonempty subset I of $\{1, \ldots, n\}$, if the restriction of f on $\mathbb{C}^I$ is not identically zero, then $\Gamma(f) \cap \mathbb{R}^I = \Gamma(f|_{\mathbb{C}^I})$.

(ii) Let $\Delta := \Delta(q, \Gamma(f))$ for some nonzero vector $q := (q_1, \ldots, q_n) \in \mathbb{R}^n$ and put $d := d(q, \Gamma(f))$. By definition, $f_\Delta = \sum_{\alpha \in \Delta} a_\alpha x^\alpha$ is a weighted homogeneous polynomial of type (q, d), i.e., we have for all $t > 0$ and all $x \in \mathbb{C}^n$,

$$f_\Delta(t^{q_1} x_1, \ldots, t^{q_n} x_n) = t^d f_\Delta(x_1, \ldots, x_n).$$

This implies the Euler relation

$$\sum_{i=1}^{n} q_i x_i \frac{\partial f_\Delta}{\partial x_i}(x) = d f_\Delta(x).$$

In particular, if $d \neq 0$ and $\nabla f_\Delta(x) = 0$, then $f_\Delta(x) = 0$.

Let $g_1, \ldots, g_p \colon \mathbb{C}^n \to \mathbb{C}$ be polynomial functions.

Definition 1.17. *The algebraic set $S := \{x \in \mathbb{C}^n \mid g_1(x) = 0, \ldots, g_p(x) = 0\}$ is non-singular if the system of of gradient vectors*

$$\nabla g_1(x), \ldots, \nabla g_p(x)$$

is $\mathbb{C}$-linearly independent for all $x \in S$.

[1] Since $\Gamma(f)$ is an integer polyhedron, we can assume that all the coordinates of q are rational numbers.

The following definition of the non-degeneracy condition is inspired from the work of Kouchnirenko (Kouchnirenko, 1976), where the case $S = \mathbb{C}^n$ was considered.

Definition 1.18. We say that the restriction of f on S is *Newton non-degenerate (at infinity)* if, and only if, for every nonempty subset $I \subset \{1, \ldots, n\}$ with $f|_{\mathbb{C}^I} \not\equiv 0$, for every (possibly empty) subset $J \subset \{j \in \{1, \ldots, p\} \mid g_j|_{\mathbb{C}^I} \not\equiv 0\}$, and for every vector $q \in \mathbb{R}^n$ with $\min_{i \in I} q_i < 0$, the following conditions hold:

(i) the set

$$\{x \in \mathbb{C}^{*n} \mid g_{j,\Delta_j}(x) = 0 \text{ for } j \in J\}$$

is a reduced smooth complete intersection variety in the torus $\mathbb{C}^{*n}$, i.e., the system of gradient vectors $\nabla g_{j,\Delta_j}(x)$ for $j \in J$ is $\mathbb{C}$-linearly independent on this variety;

(ii) if $d(q, \Gamma(f|_{\mathbb{C}^I})) < 0$, then the set

$$\{x \in \mathbb{C}^{*n} \mid f_{\Delta_0}(x) = 0 \text{ and } g_{j,\Delta_j}(x) = 0 \text{ for } j \in J\}$$

is a reduced smooth complete intersection variety in the torus $\mathbb{C}^{*n}$;

where $\Delta_0 := \Delta(q, \Gamma(f|_{\mathbb{C}^I}))$ and $\Delta_j := \Delta(q, \Gamma(g_j|_{\mathbb{C}^I}))$ for $j \in J$.

Notation 1.19. In chapter 3, we consider the above condition in the case of the variables ranging over $\mathbb{R}$ and $S = \mathbb{R}^n$.

Finally, following (Némethi & Zaharia, 1990), we introduce a set, which plays an important role in Chapter 2.

Definition 1.20. We denote $\Sigma_\infty(f|_S)$ by the set of all values $c \in \mathbb{C}$ for which there exist a nonempty set $I \subset \{1, \ldots, n\}$ with $f|_{\mathbb{C}^I} \not\equiv 0$, a (possibly empty) set $J \subset \{j \in \{1, \ldots, p\} : g_j|_{\mathbb{C}^I} \not\equiv 0\}$, a vector $q \in \mathbb{R}^n$ with $\min_{i \in I} q_i < 0$ and $d(q, \Gamma(f|_{\mathbb{C}^I})) = 0$, a point $x \in \mathbb{C}^{*I}$, and scalars $\lambda_j \in \mathbb{C}$ for $j \in J$, such that the following conditions hold:

$$\begin{aligned} c &= f_{\Delta_0}(x), \\ g_{j,\Delta_j}(x) &= 0 \text{ for } j \in J, \\ \nabla f_{\Delta_0}(x) + \sum_{j \in J} \lambda_j \nabla g_{j,\Delta_j}(x) &= 0, \end{aligned}$$

where $\Delta_0 := \Delta(q, \Gamma(f|_{\mathbb{C}^I}))$ and $\Delta_j := \Delta(q, \Gamma(g_j|_{\mathbb{C}^I}))$ for $j \in J$.

We observe that the above value $c \in \Sigma_\infty(f|_S)$ is indeed a critical value of the restriction of the polynomial f_{Δ_0} on the variety

$$\{x \in \mathbb{C}^{*I} \ : \ g_{j,\Delta_j}(x) = 0 \quad \text{for} \quad j \in J\}.$$

Hence, by the Bertini–Sard theorem, $\Sigma_\infty(f|_S)$ is a finite set provided that the restriction $f|_S$ is Newton non-degenerate at infinity.

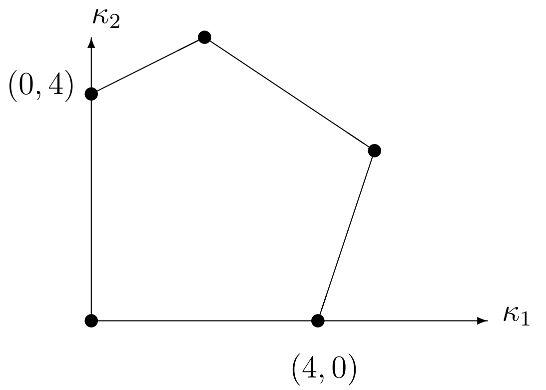
Example 1.21. The following pictures give the Newton polyhedron of polynomial functions of two variable. In detail, we have

(a) stands for a convenient polynomial function.

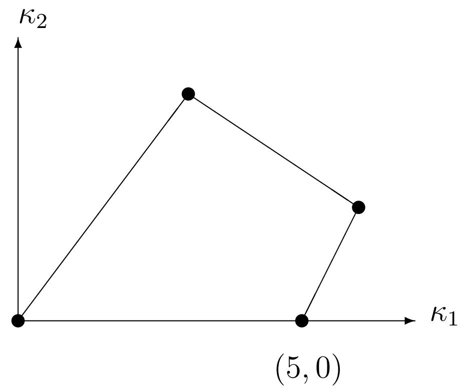
(b) shows a polynomial function which is not convenient.

(c) figures out a function which is not Newton non-degenerate in case $S = \mathbb{C}^n$.

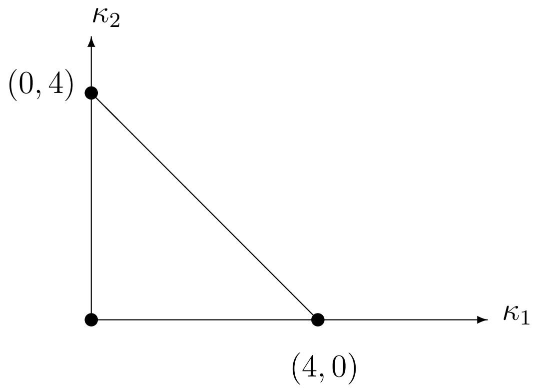
(d) illustrates a Newton polyhedron used in Chapter 3.



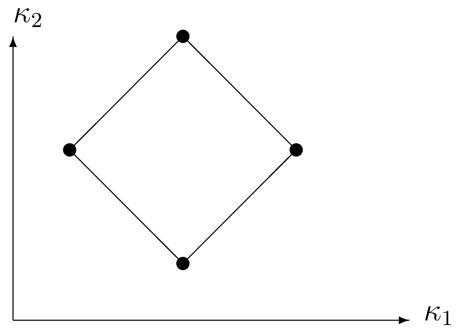
(a) $f(x, y) = x^4 + x^5y^3 + x^2y^5 + y^4$



(b) $f(x, y) = x^4y^4 - x^6y^2 + 2x^5$



(c) $f(x,y) = x^4 - 2x^2y^2 + y^4$



(d) $f(x,y) = xy^3 - 3x^5y^3 + x^3y^5 + 3x^3y$

1.3 Bertini-Sard theorem

The below knowledge of this section can be found in (Spivak, 1965).

Definition 1.22. A set M of $\mathbb{R}^n$ is called a $k-$ dimensional *manifold* if for every point $x \in M$, there is an open set U containing $x, V \subset \mathbb{R}^k$ and diffeomorphism $\varphi \colon U \to V$ such that $\varphi(U \cap M) = V$. We call (φ, U) a chart of M.

Theorem 1.23. Let M $\subset \mathbb{R}^n$ and Θ be an open cover of M. Then, there exist a collection Φ of C^∞ functions φ defined in open sets containing M with the following properties:

(i) For each $x \in M$, we have $0 \leq \varphi(x) \leq 1$.

(ii) For each $x \in M$ there is an open set V containing x such that all but finitely many $\varphi \in \Phi$ are 0 on V.

(iii) For each $x \in M$, we have $\sum_{\varphi \in \Phi} \varphi(x) = 1$.

(iv) For each $\varphi \in \Phi$, there is an open set U in Θ such that $\varphi = 0$ outside of some closed set contain in U.

A collection Φ is called *a C^∞ partition of unity* of M.

Definition 1.24. Let M, N be submanifolds of a manifold Y . M transversally with N if at every point $x \in M \cap N$, we have $T_xM + T_xN = T_xY$.

Let $f\colon A \to B$ be a smooth map of a manifold A to B.

Definition 1.25. The map f is called a locally trivial fibration if for any $b \in B$ there exists a neighbourhood U of B and a diffeomorphism (called the trivialization) from $f^{-1}(U)$ to $U \times f^{-1}(b)$ preserving the projection to U.

Theorem 1.26. (Bertini-Sard theorem) *The measure of the set of critical values of f is equal to zero.*

Chapter 2

Bifurcation Sets and Global Monodromies of Newton Non-degenerate Polynomials on Algebraic Sets

The following chapter bases on the result [BP-2] in List of Author's Related Papers.

Let $S \subset \mathbb{C}^n$ be a non-singular algebraic set and $f \colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial function. It is well-known that the restriction $f|_S \colon S \to \mathbb{C}$ of f on S is a locally trivial fibration outside a finite set $B(f|_S) \subset \mathbb{C}$. In this chapter, we give an explicit description of a finite set $T_\infty(f|_S) \subset \mathbb{C}$ such that $B(f|_S) \subset K_0(f|_S) \cup T_\infty(f|_S)$, where $K_0(f|_S)$ denotes the set of critical values of the $f|_S$. Furthermore, $T_\infty(f|_S)$ is contained in the set of critical values of certain polynomial functions provided that the $f|_S$ is Newton non-degenerate at infinity. Using these facts, we show that if $\{f_t\}_{t\in[0,1]}$ is a family of polynomials such that the Newton polyhedron at infinity of f_t is independent of t and the $f_t|_S$ is Newton non-degenerate at infinity, then the global monodromies of the $f_t|_S$ are all isomorphic.

2.1 The bifurcation set of a polynomial function

From now on, let $g_1, \ldots, g_p \colon \mathbb{C}^n \to \mathbb{C}$ be polynomial functions such that the algebraic set

$$S := \{x \in \mathbb{C}^n \mid g_1(x) = 0, \ldots, g_p(x) = 0\}$$

is non-singular.

Lemma 2.1. *There exists a real number $R_0 > 0$ such that for all $R \geqslant R_0$, the set S intersects transversally with the sphere $\mathbb{S}_R^{2n-1}$.*

Proof. We argue by contradiction. Suppose that there exist sequences $\{x^{(k)}\}_{k\in\mathbb{N}} \subset \mathbb{C}^n$ and $\{\lambda_j^{(k)}\}_{k\in\mathbb{N}} \subset \mathbb{C}, j = 1, \ldots, p+1$, such that

(a1) $\|x^{(k)}\| \to \infty$ as $k \to \infty$;

(a2) $g_j(x^{(k)}) = 0$ for all $j = 1, \ldots, p$, and all $k \in \mathbb{N}$;

(a3) $\sum_{j=1}^{p} \lambda_j^{(k)} \nabla g_j(x^{(k)}) = \lambda_{p+1}^{(k)} x^{(k)}$ for all $k \in \mathbb{N}$;

(a4) The numbers $\lambda_j^{(k)}, j = 1, \ldots, p+1$, are not all zero for all $k \in \mathbb{N}$.

By the Curve Selection Lemma at infinity (see Theorem 1.14), there exist analytic curves

$$\phi \colon (0, \epsilon) \to \mathbb{C}^n \quad \text{and} \quad \lambda_j \colon (0, \epsilon) \to \mathbb{C},\ j = 1, \ldots, p+1,$$

such that

(a5) $\|\phi(s)\| \to \infty$ as $s \to 0$;

(a6) $g_j(\phi(s)) = 0$ for all $j = 1, \ldots, p$, and all $s \in (0, \epsilon)$;

(a7) $\sum_{j=1}^{p} \lambda_j(s) \nabla g_j(\phi(s)) = \lambda_{p+1}(s)\phi(s)$ for $s \in (0, \epsilon)$;

(a8) $\lambda_j(s), j = 1, \ldots, p+1$, are not all zero for $s \in (0, \epsilon)$.

We have

$$\begin{aligned}\left\langle \frac{d\phi(s)}{ds}, \sum_{j=1}^{p} \lambda_j(s) \nabla g_j(\phi(s)) \right\rangle &= \sum_{j=1}^{p} \overline{\lambda_j(s)} \left\langle \frac{d\phi(s)}{ds}, \nabla g_j(\phi(s)) \right\rangle \\ &= \sum_{j=1}^{p} \overline{\lambda_j(s)} \frac{d}{ds}(g_j \circ \phi)(s) \\ &= 0.\end{aligned}$$

Combined with the condition (a7), this implies that

$$\overline{\lambda_{p+1}(s)} \left\langle \frac{d\phi(s)}{ds}, \phi(s) \right\rangle = 0.$$

But $\lambda_{p+1} \not\equiv 0$, which follows from the non-singularity of S and the condition (a7). Hence for all $s > 0$ small enough,

$$\left\langle \frac{d\phi(s)}{ds}, \phi(s) \right\rangle = 0$$

and so

$$\frac{d\|\phi(s)\|^2}{ds} = 2\Re \left\langle \frac{d\phi(s)}{ds}, \phi(s) \right\rangle = 0,$$

which contradicts the condition (a5). □

For the rest of this section, let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial function. It is well known that the bifurcation set $B(f|_S)$ of the restriction $f|_S\colon S \to \mathbb{C}$ contains the set $K_0(f|_S)$. Recall that we write $K_0(f|_S)$ for *the set of critical values* of the restriction of f on S, i.e.,

$$\begin{aligned}K_0(f|_S) := \{c \in \mathbb{C} \quad | \quad &\exists x \in S, \exists \lambda_j \in \mathbb{C}, j = 1, \ldots, p, \text{ such that} \\ &f(x) = c \quad \text{and} \quad \nabla f(x) + \sum_{j=1}^{p} \lambda_j \nabla g_j(x) = 0\}.\end{aligned}$$

By theorem Logical formulation of the Tarski-Seidenberg theorem (see Theorem 1.8), $K_0(f|_S)$ is a semi-algebraic set in $\mathbb{C}$ and hence, by the Bertini–Sard theorem, $K_0(f|_S)$ is a finite set.

Before formulating our first theorem, we also need the following concept (see also (Hà & Phạm, 2017, Chapter 2)).

Definition 2.2. By the *set of tangency values at infinity* of $f|_S$ we mean the set

$$\begin{aligned} T_\infty(f|_S) \ := \ & \{c \in \mathbb{C} \mid \exists \{x^{(k)}\} \subset S, \exists \{\lambda_j^{(k)}\} \subset \mathbb{C}, j = 1, \ldots, p+1, \|x^{(k)}\| \to \infty, \\ & f(x^{(k)}) \to c, \ \nabla f(x^{(k)}) + \sum_{j=1}^{p} \lambda_j^{(k)} \nabla g_j(x^{(k)}) = \lambda_{p+1}^{(k)} x^{(k)} \ \text{ for all } \ k \in \mathbb{N}\}. \end{aligned}$$

The results of Theorem 0.1 and Theorem 0.2 inspire us to introduce two following theorems.

Theorem 2.3. *$T_\infty(f|_S)$ is a finite set and the following inclusion holds*

$$B(f|_S) \ \subset \ K_0(f|_S) \cup T_\infty(f|_S). \tag{2.1}$$

Proof. In order to prove the set $T_\infty(f|_S)$ is finite, we use the *set of asymptotic critical values at infinity* of $f|_S$ (see (Jelonek, 2004; Jelonek & Kurdyka, 2005; Kurdyka et al., 2000; Rabier, 1997)):

$$\begin{aligned} K_\infty(f, S) \ := \ & \{c \in \mathbb{C} \mid \exists \{x^{(k)}\} \subset S, \|x^{(k)}\| \to \infty, f(x^{(k)}) \to c, \text{and} \\ & \|x^{(k)}\| \nu(x^{(k)}) \to 0 \ \text{ as } \ k \to \infty\}, \end{aligned}$$

where $\nu \colon \mathbb{C}^n \to \mathbb{R}$ is the *Rabier function* defined by

$$\nu(x) := \inf \left\{ \left\| \nabla f(x) + \sum_{j=1}^{p} \lambda_j \nabla g_j(x) \right\| \mid \lambda_j \in \mathbb{C}, j = 1, \ldots, p \right\}.$$

We will show that

$$T_\infty(f|_S) \ \subset \ K_\infty(f, S). \tag{2.2}$$

This, of course, implies immediately that $T_\infty(f|_S)$ is a finite set because we know from (Jelonek & Kurdyka, 2005, Theorem 3.3) that $K_\infty(f|_S)$ is a finite set.

In order to prove the inclusion (2.2), take any $c \in T_\infty(f|_S)$. By definition, there exist sequences $\{x^{(k)}\}_{k \in \mathbb{N}} \subset \mathbb{C}^n$ and $\{\lambda_j^{(k)}\}_{k \in \mathbb{N}} \subset \mathbb{C}, j = 1, \ldots, p+1$, such that

(b1) $\|x^{(k)}\| \to \infty$ as $k \to \infty$;

(b2) $f(x^{(k)}) \to c$ as $k \to \infty$;

(b3) $g_j(x^{(k)}) = 0$ for all $j = 1, \ldots, p$, and all $k \in \mathbb{N}$;

(b4) $\nabla f(x^{(k)}) + \sum_{j=1}^{p} \lambda_j^{(k)} \nabla g_j(x^{(k)}) = \lambda_{p+1}^{(k)} x^{(k)}$ for all $k \in \mathbb{N}$.

By the Curve Selection Lemma at infinity (see Theorem 1.14), there exist analytic curves

$$\phi\colon (0, \epsilon) \to \mathbb{C}^n \quad \text{and} \quad \lambda_j\colon (0, \epsilon) \to \mathbb{C},\ j = 1, \ldots, p+1,$$

such that

(b5) $\|\phi(s)\| \to \infty$ as $s \to 0$;

(b6) $f(\phi(s)) \to c$ as $s \to 0$;

(b7) $g_j(\phi(s)) = 0$ for all $j = 1, \ldots, p$, and all $s \in (0, \epsilon)$;

(b8) $\nabla f(\phi(s)) + \sum_{j=1}^{p} \lambda_j(s) \nabla g_j(\phi(s)) = \lambda_{p+1}(s)\phi(s)$ for $s \in (0, \epsilon)$.

If $\lambda_{p+1} \equiv 0$, then it is clear that $c \in K_\infty(f, S)$ and there is nothing to prove. So we may assume that λ_{p+1} is not identically zero. It follows from (b7) and (b8) that

$$\begin{aligned}
0 \not\equiv \frac{1}{2}\frac{d\|\phi(s)\|^2}{ds} &= \Re\left\langle \frac{d\phi(s)}{ds}, \phi(s) \right\rangle \\
&= \Re\left\langle \frac{d\phi(s)}{ds}, \frac{1}{\lambda_{p+1}(s)}\left[\nabla f(\phi(s)) + \sum_{j=1}^{p} \lambda_j(s)\nabla g_j(\phi(s))\right] \right\rangle \\
&= \Re\frac{1}{\overline{\lambda_{p+1}(s)}}\left[\frac{d}{ds}(f \circ \phi)(s) + \sum_{j=1}^{p} \overline{\lambda_j(s)}\frac{d}{ds}(g_j \circ \phi)(s)\right] \\
&= \Re\frac{1}{\overline{\lambda_{p+1}(s)}}\left[\frac{d}{ds}(f \circ \phi)(s)\right].
\end{aligned}$$

In particular, $f \circ \phi \not\equiv c$.

On the other hand, we may write

$$\begin{aligned}
\|\phi(s)\| &= as^{\alpha} + (\text{higher-order terms}), \\
f(\phi(s)) &= c + bs^{\beta} + (\text{higher-order terms}),
\end{aligned}$$

where $a \neq 0, b \neq 0$ and $\alpha, \beta \in \mathbb{Q}$. By the conditions (b5) and (b6) respectively, then $\alpha < 0$ and $\beta > 0$. Furthermore, a simple computation shows that

$$\mathrm{ord}_s|\lambda_{p+1}(s)| \quad \geq \quad \beta - 2\alpha.$$

It turns out from (b8) that

$$\mathrm{ord}_s\Big(\|\phi(s)\|\|\nabla f(\phi(s)) + \sum_{j=1}^{p}\lambda_j(s)\nabla g_j(\phi(s))\|\Big) \quad = \quad \mathrm{ord}_s|\lambda_{p+1}(s)| + 2\alpha \ \geq \ \beta.$$

Since $\beta > 0$, we get

$$\lim_{s\to 0^+}\Big(\|\phi(s)\|\|\nabla f(\phi(s)) + \sum_{j=1}^{p}\lambda_j(s)\nabla g_j(\phi(s))\|\Big) \quad = \quad 0.$$

Therefore, $c \in K_\infty(f, S)$, and so the inclusion (2.2) holds.

For the proof of the inclusion (2.1) we fix $c^* \in \mathbb{C} \setminus (K_0(f|_S) \cup T_\infty(f|_S))$ and D a small open disc centered at c^*, with the closure $\overline{D} \subset \mathbb{C} \setminus (K_0(f|_S) \cup T_\infty(f|_S))$. Then it is not hard to see that there exists a real number $R_0 > 0$ such that for all $c \in D$ and all $R \geqslant R_0$, the fiber $(f|_S)^{-1}(c)$ is non-singular and intersects transversally with the sphere $\mathbb{S}_R^{2n-1}$ (this is possible if D is small enough). By continuity, there exists an open neighbourhood U of $(f|_S)^{-1}(D) \cap \{x \in \mathbb{C}^n \mid \|x\| \geq R_0\}$ in $\mathbb{C}^n$ such that the vectors $\nabla f(x), \nabla g_1(x), \ldots, \nabla g_p(x)$, and x are $\mathbb{C}$-linearly independent for all $x \in U$. Therefore, we can find a smooth vector field $\mathbf{v}_1$ on U satisfying the following conditions

- $\langle \mathbf{v}_1(x), \nabla f(x)\rangle = 1$;
- $\langle \mathbf{v}_1(x), \nabla g_j(x)\rangle = 0$ for $j = 1, \ldots, p$;
- $\langle \mathbf{v}_1(x), x\rangle = 0$.

(We can construct such a vector field locally, then extend it over U by a smooth partition of unity.)

We now fix $\epsilon > 0$. Since $D \cap K_0(f, S) = \emptyset$, the vectors $\nabla f(x), \nabla g_1(x), \ldots, \nabla g_p(x)$ are $\mathbb{C}$-linearly independent for all x belonging to some open neighbourhood V of $(f|_S)^{-1}(D) \cap$

$\{x \in \mathbb{C}^n \mid \|x\| \le R_0 + \epsilon\}$ in $\mathbb{C}^n$. Consequently, there exists a smooth vector field $\mathbf{v}_2$ on V such that the following conditions hold

- $\langle \mathbf{v}_2(x), \nabla f(x) \rangle = 1$;
- $\langle \mathbf{v}_2(x), \nabla g_j(x) \rangle = 0$ for $j = 1, \ldots, p$.

(We can construct such a vector field locally, then extend it over V by a smooth partition of unity.)

Next, we fix a partition of unity θ_1 and θ_2 subordinated to the covering

$$\left\{x \in U \mid \|x\| > R_0 + \frac{\epsilon}{3}\right\} \quad \text{and} \quad \left\{x \in V \mid \|x\| < R_0 + \frac{2\epsilon}{3}\right\}$$

of $(f|S)^{-1}(D)$, and define the smooth vector field $\mathbf{v}$ on $(f|S)^{-1}(D)$ by

$$\mathbf{v} := \theta_1 \mathbf{v}_1 + \theta_2 \mathbf{v}_2.$$

Then we can see that the following conditions hold:

- $\langle \mathbf{v}(x), \nabla f(x) \rangle = 1$;
- $\langle \mathbf{v}(x), \nabla g_j(x) \rangle = 0$ for $j = 1, \ldots, p$;
- $\langle \mathbf{v}(x), x \rangle = 0$ provided that $\|x\| \ge R_0 + \epsilon$.

Finally, integrating the vector field $\mathbf{v}$ we have that the restriction $f\colon (f|_S)^{-1}(D) \to D$ is a trivial C^∞-fibration, which means that $c^* \notin B(f|_S)$. □

Remark 2.4. (i) The inclusion (2.1) provides an extension to *algebraic sets* of Theorem 1 in (Némethi & Zaharia, 1990), where the case $S = \mathbb{C}^n$ was studied.

(ii) The inclusions (2.1) and (2.2) may be strict in general, see (Păunescu & Zaharia, 1997, 2000) and (Hà & Phạm, 2008).

(iii) The proof of Theorem 2.3 also implies the following inclusion, which was proved in (Rabier, 1997; Jelonek, 2004; Jelonek & Kurdyka, 2005),

$$B(f|_S) \quad \subset \quad K_0(f|_S) \cup K_\infty(f|_S).$$

(iv) A straightforward modification shows that Lemma 3.1 and Theorem 2.3 still hold in the case where S does not have the explicit form as it was assumed; in fact, it suffices to suppose that S is a non-singular constructive subset of $\mathbb{C}^n$. As we shall not use this "improve" statement, we leave the proof as an exercise.

Under the non-degeneracy condition of Definition 1.18, we obtain the following bound of tangency values at infinity of $f|_S$ in terms of critical values of certain polynomial functions.

Theorem 2.5. *Assume that the restriction $f|_S$ of f on S is Newton non-degenerate at infinity. Then*

$$T_\infty(f|_S) \quad \subset \quad \Sigma_\infty(f|_S) \cup K_0(f|_S) \cup \{0\}.$$

Moreover, if the polynomial $f \colon \mathbb{C}^n \to \mathbb{C}$ is convenient, then $T_\infty(f|_S) = \emptyset$.

Proof. For convenience we will write g_0 instead of f.

Take arbitrarily $c \in T_\infty(g_0|_S) \setminus (K_0(g_0|_S) \cup \{0\})$. We will show that $c \in \Sigma_\infty(f|_S)$. Indeed, by definition, there exist sequences $\{x^{(k)}\}_{k\in\mathbb{N}} \subset \mathbb{C}^n$ and $\{\lambda_j^{(k)}\}_{k\in\mathbb{N}} \subset \mathbb{C}, j = 1, \ldots, p+1$, such that

(c1) $\|x^{(k)}\| \to \infty$ as $k \to \infty$;

(c2) $g_0(x^{(k)}) \to c$ as $k \to \infty$;

(c3) $g_j(x^{(k)}) = 0$ for all $j = 1, \ldots, p$, and all $k \in \mathbb{N}$;

(c4) $\nabla g_0(x^{(k)}) + \sum_{j=1}^p \lambda_j^{(k)} \nabla g_j(x^{(k)}) = \lambda_{p+1}^{(k)} x^{(k)}$ for all $k \in \mathbb{N}$.

By the Curve Selection Lemma at infinity (see Theorem 1.14), there exist analytic curves

$$\phi\colon (0,\epsilon) \to \mathbb{C}^n \quad \text{and} \quad \lambda_j\colon (0,\epsilon) \to \mathbb{C},\ j = 1,\ldots,p+1,$$

such that

(c5) $\|\phi(s)\| \to \infty$ as $s \to 0$;

(c6) $g_0(\phi(s)) \to c$ as $s \to 0$;

(c7) $g_j(\phi(s)) = 0$ for all $j = 1,\ldots,p$, and all $s \in (0,\epsilon)$;

(c8) $\nabla g_0(\phi(s)) + \sum_{j=1}^{p} \lambda_j(s)\nabla g_j(\phi(s)) = \lambda_{p+1}(s)\phi(s)$ for $s \in (0,\epsilon)$.

Put $I := \{i \mid \phi_i \not\equiv 0\}$. By the condition (c5), $I \neq \emptyset$. For $i \in I$, we can write the curve ϕ_i in terms of parameter, say

$$\phi_i(s) \;=\; x_i^0 s^{q_i} + (\text{higher-order terms}),$$

where $x_i^0 \neq 0$ and $q_i \in \mathbb{Q}$. We have $\min_{i\in I} q_i < 0$, because of the condition (c5).

If $\lambda_{p+1} \equiv 0$, then it follows from the conditions (c7) and (c8) that

$$\begin{aligned}\frac{d}{ds}(g_0 \circ \phi)(s) \;&=\; \left\langle \frac{d\phi(s)}{ds}, \nabla g_0(\phi(s))\right\rangle \;=\; -\sum_{j=1}^{p} \overline{\lambda_j(s)} \left\langle \frac{d\phi(s)}{ds}, \nabla g_j(\phi(s))\right\rangle \\ &=\; -\sum_{j=1}^{p} \overline{\lambda_j(s)} \frac{d}{ds}(g_j \circ \phi)(s) \;=\; 0.\end{aligned}$$

Consequently, $g_0(\phi(s)) = c$ for $s \in (0,\epsilon)$, and so $c \in K_0(g_0|_S)$ by (c8), which is a contradiction. Therefore, $\lambda_{p+1} \not\equiv 0$. Put $J := \{j \in \{1,\ldots,p\} \mid \lambda_j \not\equiv 0\}$. For $j \in J \cup \{p+1\}$, we can write

$$\lambda_j(s) \;=\; c_j s^{m_j} + (\text{higher-order terms}),$$

where $c_j \neq 0$ and $m_j \in \mathbb{Q}$.

Put $J_1 := \{j \in \{0\} \cup J \mid g_j|_{\mathbb{C}^I} \not\equiv 0\}$. The condition (c6) and the assumption that $c \neq 0$ together imply that $0 \in J_1$, and so $J_1 \neq \emptyset$. For each $j \in J_1$, let d_j be the minimal value of the linear function $\sum_{i\in I}\alpha_i q_i$ on $\mathbb{R}^I\cap\Gamma(g_j)$ and Δ_j be the face of $\mathbb{R}^I\cap\Gamma(g_j)$, where this linear function takes its minimum value, respectively. A simple calculation shows that

$$g_j(\phi(s)) \;=\; g_{j,\Delta_j}(x^0)s^{d_j} + (\text{higher-order terms}),$$

where $x^0 := (x_1^0,\ldots,x_n^0)$ can be understood as $x_i^0 = 1$ for $i\notin I$ because the face function g_{j,Δ_j}, which is associated with g_j and Δ_j, does not depend on the variable x_i.

The condition (c6) and the assumption that $c\neq 0$ together imply that

$$d_0 \leqslant 0 \quad\text{and}\quad d_0 g_{0,\Delta_0}(x^0) = 0. \tag{2.3}$$

Otherwise, $d_0<0$ and $g_0(\phi(s))\to+\infty$, which contradicts (c6). Furthermore, it follows from the condition (c7) that

$$g_{j,\Delta_j}(x^0) \;=\; 0 \quad\text{for all}\quad j\in J_1\setminus\{0\}. \tag{2.4}$$

On the other hand, we have for all $i\in I$ and all $j\in J_1$,

$$\frac{\partial g_j(\phi(s))}{\partial x_i} \;=\; \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0)s^{d_j-q_i} + (\text{higher-order terms}). \tag{2.5}$$

Put

$$\begin{aligned}\ell &:= \min\{m_j+d_j \mid j\in J_1\},\\ J_2 &:= \{j\in J_1 \mid \ell = m_j+d_j\}.\end{aligned}$$

Then the condition (c8) and the equality (2.5) together imply that for all $i\in I$,

$$\left(\sum_{j\in J_2}\overline{c_j}\frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0)\right)s^{\ell-q_i} + (\text{higher-order terms}) \;=\; \overline{c_{p+1}x_i^0}s^{m_{p+1}+q_i} + (\text{higher-order terms}),$$

where $c_0 := 1$ and $m_0 := 0$. Consequently, $\ell - q_i \leqslant m_{p+1}+q_i$ for all $i\in I$. Therefore,

$$\ell - m_{p+1} \;\leqslant\; 2\min_{i\in I} q_i \;<\; 0. \tag{2.6}$$

Let $I_1 := \{i \in I \mid \ell - q_i = m_{p+1} + q_i\}$. We show that the set I_1 is empty. To see this, we observe that

$$\sum_{j \in J_2} \overline{c_j} \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) \quad = \quad \begin{cases} \overline{c_{p+1}}\,\overline{x_i^0} & \text{if } i \in I_1, \\ 0 & \text{if } i \in I \setminus I_1, \\ 0 & \text{if } i \notin I, \end{cases}$$

where the last equality holds because for all $i \notin I$ and all $j \in J_2$, the polynomial g_{j,Δ_j} does not depend on the variable x_i. Consequently,

$$\begin{aligned} \sum_{i=1}^{n} \left(\sum_{j \in J_2} \overline{c_j} \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) \right) x_i^0 q_i \quad &= \quad \sum_{i \in I_1} \left(\sum_{j \in J_2} \overline{c_j} \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) \right) x_i^0 q_i \\ &= \quad \sum_{i \in I_1} \overline{c_{p+1}} |x_i^0|^2 \frac{\ell - m_{p+1}}{2}. \end{aligned}$$

The left side of the above equality can be written as

$$\begin{aligned} \sum_{i=1}^{n} \left(\sum_{j \in J_2} \overline{c_j} \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) \right) x_i^0 q_i \quad &= \quad \sum_{j \in J_2} \overline{c_j} \left(\sum_{i=1}^{n} \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) x_i^0 q_i \right) \\ &= \quad \sum_{j \in J_2} \overline{c_j} d_j g_{j,\Delta_j}(x^0), \end{aligned}$$

where the second equality follows from the Euler relation:

$$\sum_{i=1}^{n} \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) x_i^0 q_i \quad = \quad d_j g_{j,\Delta_j}(x^0) \quad \text{for } j \in J_2.$$

Therefore,

$$\sum_{i \in I_1} \overline{c_{p+1}} |x_i^0|^2 \frac{\ell - m_{p+1}}{2} \quad = \quad \sum_{j \in J_2} \overline{c_j} d_j g_{j,\Delta_j}(x^0).$$

As the left side of this equality is nonzero if $I_1 \neq \emptyset$, we get from (2.3) and (2.4) that $I_1 = \emptyset$. Thus,

$$\sum_{j \in J_2} \overline{c_j} \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) \quad = \quad 0 \quad \text{for all} \quad i = 1, \ldots, n.$$

Since the restriction of g_0 on S is Newton non-degenerate at infinity, the case where $d_0 < 0$ and $g_{0,\Delta_0} = 0$ does not occur. Thus $d_0 = 0$ and $c = g_{0,\Delta_0}(x^0) \in \Sigma_\infty(g_0|_S)$.

Finally, assume that the polynomial $f\colon \mathbb{C}^n \to \mathbb{C}$ is convenient. Then $d_0 < 0$, which is a contradiction. Hence $T_\infty(f|_S) = \emptyset$. □

For $S = \mathbb{C}^n$, the next statement was shown in (Némethi & Zaharia, 1990, Theorem 2).

Corollary 2.6. *Under the assumption of Theorem 2.5, we have*

$$B(f|_S) \quad \subset \quad \Sigma_\infty(f|_S) \cup K_0(f|_S) \cup \{0\}.$$

Moreover, if the polynomial $f\colon \mathbb{C}^n \to \mathbb{C}$ is convenient, then $B(f|_S) = K_0(f|_S)$.

Proof. This is an immediate consequence of Theorems 2.3 and 2.5 and the fact that the bifurcation set $B(f|_S)$ contains the set $K_0(f|_S)$ of critical values of the restriction $f|_S$. □

2.2 The stability of global monodromies

By the inspiration of Theorem 0.3 , we begin our works about the stability result. Recall that the (non-singular) algebraic set S is given by

$$S := \{x \in \mathbb{C}^n \mid g_1(x) = 0, \ldots, g_p(x) = 0\}.$$

In what follows, let $f\colon \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}, (t, x) \mapsto f(t, x)$, be a polynomial function. We will write $f_t(x) := f(t, x)$ and assume that for each $t \in [0, 1]$, the restriction $f_t|_S\colon S \to \mathbb{C}$ is dominant (i.e., the image set $f_t(S)$ is dense in $\mathbb{C}$).

Theorem 2.7. *Let the following conditions are satisfied:*

(i) The Newton polyhedron of f_t is independent of t;

(ii) For each $t \in [0, 1]$, the restriction $f_t|_S$ is Newton non-degenerate at infinity.

Then the global monodromies of the $f_t|_S$ are all isomorphic.

The proof of Theorem 2.7 will be divided into several steps, which, for convenience, will be called lemmas.

Lemma 2.8 (Boundedness of affine singularities)**.** *There exists a real number $r > 0$ such that*

$$K_0(f_t|_S) \quad \subset \quad D_r \quad \textit{for all} \quad t \in [0, 1].$$

Proof. Suppose the lemma were false. Then by the Curve Selection Lemma at infinity (see Theorem 1.14), there exist analytic curves

$$\phi\colon (0, \epsilon) \to \mathbb{C}^n, \quad t\colon (0, \epsilon) \to [0, 1], \quad \text{and} \quad \lambda_j\colon (0, \epsilon) \to \mathbb{C}, j = 1, \ldots, p,$$

such that

(d1) $\|\phi(s)\| \to \infty$ as $s \to 0$;

(d2) $t(s) \to t_0 \in [0, 1]$ as $s \to 0$;

(d3) $f_{t(s)}(\phi(s)) \to \infty$ as $s \to 0$;

(d4) $g_j(\phi(s)) = 0$ for all $j = 1, \ldots, p$, and all $s \in (0, \epsilon)$;

(d5) $\nabla f_{t(s)}(\phi(s)) + \sum_{j=1}^{p} \lambda_j(s) \nabla g_j(\phi(s)) = 0$ for $s \in (0, \epsilon)$.

Put $I := \{i \mid \phi_i \not\equiv 0\}$. By the condition (d1), $I \neq \emptyset$. For $i \in I$, we can write the curve ϕ_i in terms of parameter, say

$$\phi_i(s) = x_i^0 s^{q_i} + (\text{higher-order terms}),$$

where $x_i^0 \neq 0$ and $q_i \in \mathbb{Q}$. Observe that $\min_{i \in I} q_i < 0$ because of the condition (d1).

Recall from our assumptions that the Newton polyhedron $\Gamma(f_t)$ of f_t does not depend on t. By the condition (d3), $\mathbb{R}^I \cap \Gamma(f_t) \neq \emptyset$. Let d_0 be the minimal value of the linear function $\sum_{i\in I}\alpha_i q_i$ on $\mathbb{R}^I\cap\Gamma(f_t)$ and Δ_0 be the face of $\mathbb{R}^I\cap\Gamma(f_t)$ where this linear function takes its minimum value. We can write

$$\begin{aligned} f_{t(s)}(\phi(s)) &= f_{t_0,\Delta_0}(x^0)s^{d_0} + (\text{higher-order terms}), \\ \frac{\partial f_{t(s)}}{\partial x_i}(\phi(s)) &= \frac{\partial f_{t_0,\Delta_0}}{\partial x_i}(x^0)s^{d_0-q_i} + (\text{higher-order terms}) \quad \text{for all } i\in I, \end{aligned}$$

where $x^0 := (x^0_1,\dots,x^0_n)$ with $x^0_i = 1$ for $i\notin I$ and f_{t_0,Δ_0} denotes the face function of f_{t_0} with respect to the face Δ_0. By the condition (d3), $d_0 < 0$. Furthermore, for $i\notin I$, the function f_{t_0,Δ_0} does not depend on the variable x_i, and so

$$\frac{\partial f_{t_0,\Delta_0}}{\partial x_i}(x^0) \;=\; 0 \quad \text{for all} \quad i\notin I. \tag{2.7}$$

Put $J := \{j\in\{1,\dots,p\} \mid \lambda_j \not\equiv 0\}$. If $J=\emptyset$, then from the condition (d5) we deduce for all $i\in I$ that $\frac{\partial f_{t(s)}}{\partial x_i}(\phi(s)) = 0$, and hence that $\frac{\partial f_{t_0,\Delta_0}}{\partial x_i}(x^0) = 0$. It turns out from (2.7), the Euler relation, and the inequality $d_0<0$ that $f_{t_0,\Delta_0}(x^0)=0$, which contradicts the non-degeneracy condition. Therefore, $J\neq\emptyset$. For $j\in J$, we can write

$$\lambda_j(s) = c_j s^{m_j} + (\text{higher-order terms}),$$

where $c_j\neq 0$ and $m_j\in\mathbb{Q}$.

Put $J_1 := \{j\in J \mid g_j|_{\mathbb{C}^I}\not\equiv 0\}$. If $J_1=\emptyset$, then

$$\frac{\partial g_j}{\partial x_i}(\phi(s)) \;\equiv\; 0 \quad \text{for all} \quad i\in I \text{ and all } j\in J.$$

We deduce from the condition (d5) that

$$\frac{\partial f_{t(s)}}{\partial x_i}(\phi(s)) \;\equiv\; 0 \quad \text{for all} \quad i\in I.$$

Consequently,

$$\frac{\partial f_{t_0,\Delta_0}}{\partial x_i}(x^0) \;=\; 0 \quad \text{for all} \quad i\in I.$$

It follows from (2.7), the Euler relation, and the inequality $d_0<0$ that $f_{t_0,\Delta_0}(x^0)=0$, which contradicts the non-degeneracy condition. Hence $J_1\neq\emptyset$. For each $j\in J_1$, let d_j

be the minimal value of the linear function $\sum_{i\in I}\alpha_i q_i$ on $\mathbb{R}^I \cap \Gamma(g_j)$ and Δ_j be the face of $\mathbb{R}^I \cap \Gamma(g_j)$ where this linear function takes its minimum value. We can write

$$g_j(\phi(s)) \;=\; g_{j,\Delta_j}(x^0)s^{d_j} + (\text{higher-order terms}),$$

where g_{j,Δ_j} is the face function of g_j with respect to the face Δ_j. By the condition (d4), then

$$g_{j,\Delta_j}(x^0) \;=\; 0 \quad \text{for all} \quad j \in J_1. \tag{2.8}$$

On the other hand, for $i \in I$ and $j \in J_1$,

$$\frac{\partial g_j}{\partial x_i}(\phi(s)) \;=\; \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0)s^{d_j - q_i} + (\text{higher-order terms}).$$

For $i \notin I$ and $j \in J_1$, the function g_{j,Δ_j} does not depend on the variable x_i, and hence,

$$\frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) \;=\; 0. \tag{2.9}$$

Let

$$\ell \;:=\; \min_{j\in J_1}(m_j + d_j),$$
$$J_2 \;:=\; \{j \in J_1 \mid \ell \;=\; m_j + d_j\}.$$

The condition (d5) implies that for all $i \in I$,

$$\frac{\partial f_{t_0,\Delta_0}}{\partial x_i}(x^0)s^{d_0-q_i} + (\text{higher-order terms}) + \sum_{j\in J_2}\overline{c_j}\frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0)s^{\ell-q_i} + (\text{higher-order terms}) \;=\; 0. \tag{2.10}$$

There are three cases to be considered.

Case 1: $\ell \;=\; d_0$

We deduce from (2.7), (2.9) and (2.10) that

$$\frac{\partial f_{t_0,\Delta_0}}{\partial x_i}(x^0) + \sum_{j\in J_2}\overline{c_j}\frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) \;=\; 0 \quad \text{for} \quad i = 1, \ldots, n.$$

Consequently,

$$\begin{aligned}
0 &= \sum_{i=1}^{n} q_i x_i^0 \frac{\partial f_{t_0,\Delta_0}}{\partial x_i}(x^0) + \sum_{i=1}^{n} \sum_{j \in J_2} \overline{c_j} q_i x_i^0 \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) \\
&= \sum_{i=1}^{n} q_i x_i^0 \frac{\partial f_{t_0,\Delta_0}}{\partial x_i}(x^0) + \sum_{j \in J_2} \overline{c_j} \sum_{i=1}^{n} q_i x_i^0 \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) \\
&= d_0 f_{t_0,\Delta_0}(x^0) + \sum_{j \in J_2} \overline{c_j} d_j g_{j,\Delta_j}(x^0) \\
&= d_0 f_{t_0,\Delta_0}(x^0),
\end{aligned}$$

where the last equality follows from (2.8). Since $d_0 < 0$, we get $f_{t_0,\Delta_0}(x^0) = 0$, which contradicts the non-degeneracy condition.

Case 2: $\ell > d_0$

By (2.7) and (2.10), we have

$$\frac{\partial f_{t_0,\Delta_0}}{\partial x_i}(x^0) = 0 \quad \text{for} \quad i = 1, \ldots, n.$$

Then by a similar argument to that in Case 1, we also get a contradiction.

Case 3: $\ell < d_0$

By (2.9) and (2.10), we obtain

$$\sum_{j \in J_2} \overline{c_j} \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) = 0 \quad \text{for} \quad i = 1, \ldots, n.$$

This fact and (2.8) combined give a contradiction with the non-degeneracy condition. □

Lemma 2.9 (Boundedness of singularities at infinity)**.** *There exists a real number $r > 0$ such that*

$$\Sigma_\infty(f_t|_S) \subset D_r \quad \textit{for all} \quad t \in [0, 1].$$

Proof. Suppose the assertion of the lemma is false. By the Curve Selection Lemma at infinity (see Theorem 1.14), we can find a nonempty subset I of $\{1, \ldots, n\}$ with $f_t|_{\mathbb{C}^I} \not\equiv 0$, a (possibly empty) subset J of $\{j \in \{1, \ldots, p\} \mid g_j|_{\mathbb{C}^I} \not\equiv 0\}$, faces Δ_0 of $\Gamma(f_t|_{\mathbb{C}^I})$ and Δ_j of $\Gamma(g_j|_{\mathbb{C}^I})$ for $j \in J$, and analytic curves

$$\phi\colon (0, \epsilon) \to (\mathbb{C}^*)^I, \quad t\colon (0, \epsilon) \to [0, 1], \quad \text{and} \quad \lambda_j\colon (0, \epsilon) \to \mathbb{C}, j \in J,$$

such that the following conditions hold

(e1) $\|\phi(s)\| \to \infty$ as $s \to 0$;

(e2) $t(s) \to t_0 \in [0, 1]$ as $s \to 0$;

(e3) $f_{t(s),\Delta_0}(\phi(s)) \to \infty$ as $s \to 0$;

(e4) $g_{j,\Delta_j}(\phi(s)) = 0$ for all $j \in J$ and all $s \in (0, \epsilon)$;

(e5) $\nabla f_{t(s),\Delta_0}(\phi(s)) + \sum_{j \in J} \lambda_j(s) \nabla g_{j,\Delta_j}(\phi(s)) = 0$ for all $s \in (0, \epsilon)$.

Then by a similar argument to that given in the proof of Lemma 2.8, with f_t and g_j replaced by f_{t,Δ_0} and g_{j,Δ_j} respectively, we also reach a contradiction. The details are left to the reader. □

Lemma 2.10 (Transversality in the neighbourhood of infinity)**.** *Let r be a positive real number such that the conclusions of Lemmas 2.8 and 2.9 are fulfilled. Then there exists a real number $R_0 > 0$ such that for all $t \in [0, 1]$, all $R \geqslant R_0$ and all $c \in \mathbb{S}^1_r$, we have the fiber $(f_t|_S)^{-1}(c)$ intersects transversally with the sphere $\mathbb{S}^{2n-1}_R$.*

Proof. If the assertion is not true, then by the Curve Selection Lemma at infinity (see Theorem 1.14), there exist $t_0 \in [0, 1]$, $c \in \mathbb{S}^1_r$ and analytic curves

$$\phi\colon (0, \epsilon) \to \mathbb{C}^n, \quad t\colon (0, \epsilon) \to [0, 1], \quad \text{and} \quad \lambda_j\colon (0, \epsilon) \to \mathbb{C}, j = 0, 1, \ldots, p + 1,$$

satisfying the following conditions

(f1) $\|\phi(s)\| \to \infty$ as $s \to 0$;

(f2) $t(s) \to t_0$ as $s \to 0$;

(f3) $f_{t(s)}(\phi(s)) \to c$ as $s \to 0$;

(f4) $g_j(\phi(s)) = 0$ for all $j = 1, \ldots, p$, and all $s \in (0, \epsilon)$;

(f5) $\lambda_0(s)\nabla f_{t(s)}(\phi(s)) + \sum_{j=1}^{p} \lambda_j(s)\nabla g_j(\phi(s)) = \lambda_{p+1}(s)\phi(s)$ for all $s \in (0, \epsilon)$.

Put $I := \{i \mid \phi_i \not\equiv 0\}$. By the condition (f1), $I \neq \emptyset$. For $i \in I$, we can write the curve ϕ_i in terms of parameter, say

$$\phi_i(s) \quad = \quad x_i^0 s^{q_i} + (\text{higher-order terms}),$$

where $x_i^0 \neq 0$ and $q_i \in \mathbb{Q}$. Observe that $\min_{i \in I} q_i < 0$, because of the condition (f1).

By the condition (f3) and the fact that $|c| = r > 0$, we have $\mathbb{R}^I \cap \Gamma(f_t) \neq \emptyset$. Let d_0 be the minimal value of the linear function $\sum_{i \in I} \alpha_i q_i$ on $\mathbb{R}^I \cap \Gamma(f_t)$ and Δ_0 be the face of $\mathbb{R}^I \cap \Gamma(f_t)$ where this linear function takes its minimum value. As the Newton polyhedron $\Gamma(f_t)$ of f_t does not depend on t, we can write

$$\begin{aligned} f_{t(s)}(\phi(s)) \quad &= \quad f_{t_0,\Delta_0}(x^0)s^{d_0} + (\text{higher-order terms}), \\ \frac{\partial f_{t(s)}}{\partial x_i}(\phi(s)) \quad &= \quad \frac{\partial f_{t_0,\Delta_0}}{\partial x_i}(x^0)s^{d_0 - q_i} + (\text{higher-order terms}) \qquad \text{for } i \in I, \end{aligned}$$

where $x^0 := (x_1^0, \ldots, x_n^0)$ with $x_i^0 = 1$ for $i \notin I$. The condition (f3) and the fact that $|c| = r > 0$ together imply that

$$d_0 \ \leqslant \ 0 \qquad \text{and} \qquad d_0 f_{t_0,\Delta_0}(x^0) \ = \ 0. \tag{2.11}$$

Furthermore, for $i \notin I$, the function f_{t_0,Δ_0} does not depend on the variable x_i, and so

$$\frac{\partial f_{t_0,\Delta_0}}{\partial x_i}(x^0) \quad = \quad 0 \qquad \text{for all} \qquad i \notin I. \tag{2.12}$$

On the other hand, we deduce from the condition (f5), Lemmas 3.1 and 2.8 that $\lambda_0 \not\equiv 0$ and $\lambda_{p+1} \not\equiv 0$ (perhaps after reducing ϵ). Replacing λ_j by $\frac{\lambda_j}{\lambda_0}$ if necessary, we may assume that $\lambda_0 \equiv 1$. Put $J := \{j \in \{1, \dots, p\} \mid \lambda_j \not\equiv 0\}$. For $j \in J \cup \{p+1\}$, we can write

$$\lambda_j(s) \;=\; c_j s^{m_j} + (\text{higher-order terms}),$$

where $c_j \neq 0$ and $m_j \in \mathbb{Q}$.

Put $J_1 := \{j \in J \mid g_j|_{\mathbb{C}^I} \not\equiv 0\}$. We only consider the case $J_1 \neq \emptyset$ because the case $J_1 = \emptyset$ is handled similarly. For each $j \in J_1$, let d_j be the minimal value of the linear function $\sum_{i \in I} \alpha_i q_i$ on $\mathbb{R}^I \cap \Gamma(g_j)$ and Δ_j be the face of $\mathbb{R}^I \cap \Gamma(g_j)$ where this linear function takes its minimum value. We can write

$$g_j(\phi(s)) \;=\; g_{j,\Delta_j}(x^0) s^{d_j} + (\text{higher-order terms}).$$

By the condition (f4), then

$$g_{j,\Delta_j}(x^0) \;=\; 0 \quad \text{for all} \quad j \in J_1, \tag{2.13}$$

On the other hand, for $i \in I$ and $j \in J_1$,

$$\frac{\partial g_j}{\partial x_i}(\phi(s)) \;=\; \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) s^{d_j - q_i} + (\text{higher-order terms}).$$

For $i \notin I$ and $j \in J_1$, the function g_{j,Δ_j} does not depend on the variable x_i, and hence,

$$\frac{\partial g_{j,\Delta_j}}{\partial x_i}(x_0) \;=\; 0. \tag{2.14}$$

Let $\ell := \min_{j \in J_1}(m_j + d_j)$ and $J_2 := \{j \in J_1 \mid m_j + d_j = \ell\}$. From the condition (f5), for $i \in I$ we have

$$\begin{aligned} \frac{\partial f_{t_0,\Delta_0}}{\partial x_i}(x^0) s^{d_0 - q_i} \;&+\; (\text{higher-order terms}) + \sum_{j \in J_2} \overline{c_j} \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) s^{\ell - q_i} + (\text{higher-order terms}) \\ &=\; \overline{c_{p+1}}\,\overline{x_i^0} s^{m_{p+1} + q_i} + (\text{higher-order terms}). \end{aligned} \tag{2.15}$$

There are three cases to be considered.

Case 1: $\ell = d_0$

From (2.15) we have $d_0 - q_i \leqslant m_{p+1} + q_i$ for all $i \in I$. Therefore

$$d_0 - m_{p+1} \ \leqslant \ 2 \min_{i \in I} q_i \ < \ 0.$$

Put $I_1 := \{i \in I \mid d_0 - q_i = m_{p+1} + q_i\}$. Hence, $i \in I \setminus I_1$ if, and only if,

$$\frac{\partial f_{t_0,\Delta_0}}{\partial x_i}(x^0) + \sum_{j \in J_2} \overline{c_j} \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) = 0,$$

and in this case $d_0 - q_i < m_{p+1} + q_i$.

If $I_1 = \emptyset$, then

$$\frac{\partial f_{t_0,\Delta_0}}{\partial x_i}(x^0) + \sum_{j \in J_2} \overline{c_j} \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) \ = \ 0 \quad \text{for all} \quad i = 1, \ldots, n.$$

Hence, the non-degeneracy condition, (2.11) and (2.13) together imply that $d_0 = 0$. Consequently, by the condition (f2), $c = f_{t_0,\Delta_0}(x^0) \in \Sigma_\infty(f_{t_0}|_S)$, which contradicts our assumption.

If $I_1 \neq \emptyset$, then from (2.15) we have for all $i \in I_1$,

$$\begin{aligned} \frac{\partial f_{t_0,\Delta_0}}{\partial x_i}(x^0) + \sum_{j \in J_2} \overline{c_j} \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) &= \overline{c_{p+1} x_i^0}, \\ d_0 - m_{p+1} &= 2q_i. \end{aligned}$$

This, together with the Euler relation, (2.11), (2.12), (2.13) and (2.14), yield

$$\begin{aligned} 0 &= d_0 f_{t_0,\Delta_0}(x^0) + \sum_{j \in J_2} \overline{c_j} d_j g_{j,\Delta_j}(x^0) \\ &= \sum_{i=1}^n q_i x_i^0 \frac{\partial f_{t_0,\Delta_0}}{\partial x_i}(x^0) + \sum_{j \in J_2} \sum_{i=1}^n \overline{c_j} q_i x_i^0 \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) \\ &= \sum_{i=1}^n q_i x_i^0 \left(\frac{\partial f_{t_0,\Delta_0}}{\partial x_i}(x^0) + \sum_{j \in J_2} \overline{c_j} \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) \right) \\ &= \sum_{i \in I_1} q_i x_i^0 \left(\frac{\partial f_{t_0,\Delta_0}}{\partial x_i}(x^0) + \sum_{j \in J_2} \overline{c_j} \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) \right) \\ &= \sum_{i \in I_1} |x^0|_i^2 \frac{d_0 - m_{p+1}}{2} \overline{c_{p+1}} \ \neq \ 0, \end{aligned}$$

which is impossible.

Case 2: $\ell > d_0$

The same argument as in Case 1 yields a contradiction.

Case 3: $\ell < d_0$

From (2.15) we have $\ell - q_i \leqslant m_{p+1} + q_i$ for all $i \in I$. Therefore

$$\ell - m_{p+1} \leqslant 2 \min_{i \in I} q_i < 0.$$

Put $I_2 := \{i \in I \mid \ell - q_i = m_{p+1} + q_i\}$. Hence, $i \in I \setminus I_2$ if, and only if,

$$\sum_{j \in J_2} \overline{c_j} \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) = 0,$$

and in this case $\ell - q_i < m_{p+1} + q_i$.

If $I_2 = \emptyset$, then

$$\sum_{j \in J_2} \overline{c_j} \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) = 0 \quad \text{for all} \quad i = 1, \ldots, n,$$

which, together with (2.13), leads to a contradiction with the non-degeneracy condition.

If $I_2 \neq \emptyset$, then from (2.15) we have for all $i \in I_2$,

$$\begin{aligned} \sum_{j \in J_2} \overline{c_j} \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) &= \overline{c_{p+1}} \overline{x_i^0}, \\ \ell - m_{p+1} &= 2q_i. \end{aligned}$$

This, together with the Euler relation and (2.13), yields

$$\begin{aligned}
0 &= \sum_{j\in J_2} \overline{c_j} d_j g_{j,\Delta_j}(x^0) \\
&= \sum_{j\in J_2}\sum_{i=1}^{n} \overline{c_j} q_i x_i^0 \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) \\
&= \sum_{i=1}^{n}\sum_{j\in J_2} \overline{c_j} q_i x_i^0 \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) \\
&= \sum_{i\in I_2} q_i x_i^0 \left(\sum_{j\in J_2} \overline{c_j} \frac{\partial g_{j,\Delta_j}}{\partial x_i}(x^0) \right) \\
&= \sum_{i\in I_2} |x^0|_i^2 \frac{\ell - m_{p+1}}{2} \overline{c_{p+1}} \ \neq \ 0,
\end{aligned}$$

which is impossible. □

We now can complete the proof of Theorem 2.7.

Proof of Theorem 2.7. Let r and R_0 be the positive real numbers such that the conclusions of Lemmas 3.1, 2.8, 2.9 and 2.10 are fulfilled. Put

$$X \ := \ \{(t,x) \in [0,1]\times S \mid f(t,x) \in \mathbb{S}_r^1\}.$$

By Corollary 2.6, then $B(f_t|_S) \subset D_r$ for all $t \in [0,1]$. Furthermore, for all $(t,x) \in X \cap \{\|x\| \geqslant R_0\}$, the vectors $\nabla f_t(x), \nabla g_1(x), \ldots, \nabla g_p(x)$, and $\overline{x}$ are $\mathbb{C}$-linearly independent. Therefore, we can find a smooth map $\mathbf{v}_1 \colon X \cap \{\|x\| \geqslant R_0\} \to \mathbb{C}^n, (t,x) \mapsto \mathbf{v}_1(t,x)$, satisfying the following conditions

- $\langle \mathbf{v}_1(t,x), \nabla f_t(x)\rangle = -\frac{\partial f_t}{\partial t}(x)$;
- $\langle \mathbf{v}_1(t,x), \nabla g_j(x)\rangle = 0$ for $j = 1, \ldots, p$;
- $\langle \mathbf{v}_1(t,x), x\rangle = 0$.

We take arbitrary (but fixed) $\epsilon > 0$. Since $\mathbb{S}_r^1 \cap K_0(f_t|_S) = \emptyset$ for all $t \in [0,1]$, the vectors $\nabla f_t(x), \nabla g_1(x), \ldots, \nabla g_p(x)$ are $\mathbb{C}$-linearly independent for all $(t,x) \in X \cap \{\|x\| \leqslant R_0 + \epsilon\}$.

Consequently, there exists a smooth map $\mathbf{v}_2\colon X \cap \{\|x\| \leqslant R_0 + \epsilon\} \to \mathbb{C}^n, (t,x) \mapsto \mathbf{v}_2(t,x)$, such that the following conditions hold

- $\langle \mathbf{v}_2(t,x), \nabla f_t(x)\rangle = -\frac{\partial f_t}{\partial t}(x)$;
- $\langle \mathbf{v}_2(t,x), \nabla g_j(x)\rangle = 0$ for $j = 1, \ldots, p$.

Next, we fix a partition of unity θ_1 and θ_2 subordinated to the covering

$$\left\{(t,x) \in X \mid \|x\| > R_0 + \frac{\epsilon}{3}\right\} \quad \text{and} \quad \left\{(t,x) \in X \mid \|x\| < R_0 + \frac{2\epsilon}{3}\right\}$$

of X, and define the smooth map $\mathbf{v}\colon X \to \mathbb{C}^n, (t,x) \mapsto \mathbf{v}(t,x)$, by

$$\mathbf{v} := \theta_1 \mathbf{v}_1 + \theta_2 \mathbf{v}_2.$$

Then we can see that the following conditions hold:

- $\langle \mathbf{v}(t,x), \nabla f_t(x)\rangle = -\frac{\partial f_t}{\partial t}(x)$;
- $\langle \mathbf{v}(t,x), \nabla g_j(x)\rangle = 0$ for $j = 1, \ldots, p$;
- $\langle \mathbf{v}(t,x), x\rangle = 0$ provided that $\|x\| \geqslant R_0 + \epsilon$.

Finally, we define the smooth vector field $\mathbf{w}$ on X by

$$\mathbf{w}(t,x) := \frac{\partial}{\partial t} + \mathbf{v}_1(t,x)\frac{\partial}{\partial x_1} + \cdots + \mathbf{v}_n(t,x)\frac{\partial}{\partial x_n},$$

where $\mathbf{v}_i$ are the coordinates of the map $\mathbf{v}$, and then integrating this vector field we can see that for each $t \in [0,1]$, there exists a C^∞-diffeomorphism

$$\Phi_t\colon f_0^{-1}(\mathbb{S}^1_r) \cap S \to f_t^{-1}(\mathbb{S}^1_r) \cap S,$$

which makes the following diagram commutes

$$\begin{array}{ccc} f_0^{-1}(\mathbb{S}^1_r) \cap S & \xrightarrow{\Phi_t} & f_t^{-1}(\mathbb{S}^1_r) \cap S \\ {\scriptstyle f_0}\big\downarrow & & {\scriptstyle f_t}\big\downarrow \\ \mathbb{S}^1_r & \xrightarrow{\text{id}} & \mathbb{S}^1_r \end{array}$$

where id denotes the identity map. The proof is completed. □

In summary, this chapter establishes the following main results:

- The bifurcation set and the monodromy of a complex polynomial function f restricting to a non-singular algebraic set S in terms of its Newton polyhedron where $f|_S$ is Neton non-degenerate at infinity.

- The global monodromy of a family polynomial $\{f_t\}$ restricting on an algebraic set in which the Newton polyhedrons of $\{f_t\}$ are independent from t and satisfy the non-degeneracy condition.

Chapter 3

Compactness criteria for real algebraic set and Newton polyhedron

The following chapter bases on the result [BP-1] in List of Author's Related Papers.

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial and $\mathcal{Z}(f)$ its zero set. In this chapter, in terms of the so-called Newton polyhedron of f, we present a necessary criterion and a sufficient condition for the compactness of $\mathcal{Z}(f)$. From this we derive necessary and sufficient criteria for the stable compactness of $\mathcal{Z}(f)$.

3.1 The compactness of an algebraic set.

From now on let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a nonconstant polynomial in $n \geq 2$ variables and let $\mathcal{Z}(f)$ be its zero set:

$$\mathcal{Z}(f) := \{x \in \mathbb{R}^n \mid f(x) = 0\}.$$

The results of Theorem 0.4 and 0.5 are the roots of the following works which focus on the case $n \geq 2$.

Lemma 3.1. *If $\mathcal{Z}(f)$ is compact, then f is bounded either from below or from above.*

Proof. Suppose the assertion of the lemma is false. We have

$$\lim_{R\to+\infty} \min_{\|x\|=R} f(x) = -\infty \quad \text{and} \quad \lim_{R\to+\infty} \max_{\|x\|=R} f(x) = +\infty.$$

Then, for R sufficiently large, there exist $a, b \in \mathbb{R}^n$ with $\|a\| = \|b\| = R$ such that $f(a) < 0 < f(b)$. Furthermore, since $\mathcal{Z}(f)$ is compact, we may assume that

$$\mathcal{Z}(f) \subset \{x \in \mathbb{R}^n \mid \|x\| < R\},$$

after perhaps increasing R.

On the other hand, the sphere $\mathbb{S}_R^{n-1} := \{x \in \mathbb{R}^n \mid \|x\| = R\}$ is path-connected (note that $n \geq 2$). Hence, there is a continuous

$$\phi\colon [0,1] \to \mathbb{S}_R^{n-1}, \quad t \mapsto \phi(t),$$

such that $\phi(0) = a$ and $\phi(1) = b$. Consequently, the composition function $f \circ \phi\colon [0,1] \to \mathbb{R}$ is continuous and satisfies

$$(f \circ \phi)(0) \times (f \circ \phi)(1) = f(a) \times f(b) < 0.$$

Thanks to the mean value theorem, we can find $t_0 \in (0,1)$ such that $(f \circ \phi)(t_0) = 0$, a contradiction. □

Lemma 3.2. *Assume that f is bounded from below and its zero set $\mathcal{Z}(f)$ is compact. Then, the sub-level set $\{x \in \mathbb{R}^n \mid f(x) \leq 0\}$ is compact.*

Proof. In order to obtain a contradiction, suppose that $\{x \in \mathbb{R}^n \mid f(x) \leq 0\}$ is not compact. Then there exists a sequence $\{a^k\}_{k\geq 1} \subset \mathbb{R}^n$ such that

$$\lim_{k\to\infty} \|a^k\| = +\infty \quad \text{and} \quad f(a^k) \leq 0 \ \text{ for all } k.$$

Let b^k be an optimal solution of the problem

$$\max_{\|x\| \,=\, \|a^k\|} f(x).$$

Since f is bounded from below, it cannot be bounded from above. In particular,

$$\lim_{k\to\infty} f(b^k) = +\infty.$$

Therefore, for all k sufficiently large,

$$f(a^k) \times f(b^k) \leq 0.$$

As in the proof of Lemma 3.1, we can find $c^k \in \mathbb{R}^n$ with $\|c^k\| = \|a^k\| = \|b^k\|$ such that $f(c^k) = 0$, which contradicts the compactness of $\mathcal{Z}(f)$. □

The following is a generalization for the necessary criterion for compactness of real algebraic sets

Theorem 3.3. *Suppose that $\mathcal{Z}(f)$ is compact. The following assertions hold true:*

(i) $f|_{\mathbb{R}^J} \not\equiv 0$ *for all* $J \subset \{1, \ldots, n\}$,

(ii) One of the following statements holds

(ii1) f *is bounded from below and* $f_\Delta \geq 0$ *on* $\mathbb{R}^n$ *for all* $\Delta \in \Gamma_\infty(f)$.

(ii2) f *is bounded from above and* $f_\Delta \leq 0$ *on* $\mathbb{R}^n$ *for all* $\Delta \in \Gamma_\infty(f)$.

Proof. (i) This is obvious.

(ii) By Lemma 3.1, f is bounded either from below or from above.

Assume that f is bounded from below; the case f is bounded from above is treated similarly. Take any $\Delta \in \Gamma_\infty(f)$. We will show that $f_\Delta \geq 0$ on $\mathbb{R}^n$. In fact, since f is continuous, it suffices to prove that $f_\Delta \geq 0$ on $(\mathbb{R} \setminus \{0\})^n$. Suppose to the contrary that there is a point $x^0 \in (\mathbb{R} \setminus \{0\})^n$ such that $f_\Delta(x^0) < 0$. By definition, there exists a vector $q \in \mathbb{R}^n$ with $\min_{j=1,\ldots,n} q_j < 0$ such that $\Delta = \Delta(q, \Gamma(f))$. Define the monomial curve

$$\phi\colon (0, +\infty) \to \mathbb{R}^n, \quad t \mapsto (x_1^0 t^{q_1}, \ldots, x_n^0 t^{q_n}).$$

Then $\|\phi(t)\| \to +\infty$ as $t \to 0^+$. Furthermore, a simple calculation shows that for all $t > 0$ small enough we have

$$f(\phi(t)) = f_\Delta(x^0) t^d + \text{ higher-order terms},$$

where $d := d(q, \Gamma(f))$. Since $f_\Delta(x^0) < 0$ it follows that $f(\phi(t)) < 0$ for all $t > 0$ sufficiently small. Hence, the sub-level set $\{x \in \mathbb{R}^n \mid f(x) \leq 0\}$ is not compact, which contradicts Lemma 3.2. □

The following remark shows that the converse of Theorem 3.3 does not hold.

Remark 3.4. Let $n = 2$ and consider the polynomial

$$f(x_1, x_2) := (x_1 - x_2)^2.$$

By definition, the Newton polyhedron $\Gamma(f)$ is a segment joining the two points $(2, 0)$ and $(0, 2)$, and so the Newton boundary $\Gamma_\infty(f)$ is the union of the faces:

$$\Delta_1 := \{(2, 0)\}, \quad \Delta_2 := \{(0, 2)\}, \quad \text{and} \quad \Delta_3 := \{(1-t)(2, 0) + t(0, 2) \mid 0 \leq t \leq 1\}.$$

Clearly, the polynomials $f_{\Delta_1}(x_1, x_2) = x_1^2$, $f_{\Delta_2}(x_1, x_2) = x_2^2$, and $f_{\Delta_3}(x_1, x_2) = (x_1 - x_2)^2$ are all non-negative on $\mathbb{R}^n$. However, $\mathcal{Z}(f) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = x_2\}$ is not compact. On the other hand, we have the following statement, which provides a sufficient condition for compactness of real algebraic sets.

Theorem 3.5. *Suppose the following conditions:*

(i) $f|_{\mathbb{R}^J} \not\equiv 0$ *for all* $J \subset \{1, \ldots, n\}$,

(ii) One of the following statements holds

(ii1) $f_\Delta > 0$ *on* $(\mathbb{R} \setminus \{0\})^n$ *for all* $\Delta \in \Gamma_\infty(f)$.

(ii2) $f_\Delta < 0$ *on* $(\mathbb{R} \setminus \{0\})^n$ *for all* $\Delta \in \Gamma_\infty(f)$.

Then $\mathcal{Z}(f)$ *is compact.*

Proof. Assume the assertion of the theorem is false, i.e., there exists a sequence of points $\{a^k\}_{k\geq 1} \subset \mathcal{Z}(f)$ such that $\lim_{k\to\infty} \|a^k\| = +\infty$. By the Curve Selection Lemma at infinity (see Theorem 1.14), there exists an analytic curve

$$\phi\colon (0, \epsilon) \to \mathbb{R}^n, \quad t \mapsto (\phi_1(t), \ldots, \phi_n(t)),$$

such that

(a) $\|\phi(t)\| \to +\infty$ as $t \to 0^+$,

(b) $f(\phi(t)) = 0$ for $t \in (0, \epsilon)$.

Let $J := \{j \mid \phi_j \not\equiv 0\} \subseteq \{1, \ldots, n\}$. Recall that $\mathbb{R}^J := \{\alpha \in \mathbb{R}^n \mid \alpha_j = 0 \text{ for } j \notin J\}$. By Condition (a), $J \neq \emptyset$. For $j \in J$, we can expand the function ϕ_j in terms of parameter, say

$$\phi_j(t) = x_j^0 t^{q_j} + \text{ higher-order terms},$$

where $x_j^0 \neq 0$ and $q_j \in \mathbb{Q}$. By Condition (a) again, we obtain $\min_{j\in J} q_j < 0$.

Note that the curve ϕ lies in $\mathbb{R}^J \cap \mathcal{Z}(f)$. Hence, by the assumption (i), the restriction of f on $\mathbb{R}^J$ is not constant; in particular, the polyhedron $\Gamma(f|_{\mathbb{R}^J})$ is nonempty and different from $\{0\}$. Let d be the minimal value of the linear function $\sum_{j\in J} q_j \alpha_j$ on $\Gamma(f|_{\mathbb{R}^J})$ and let Δ be the maximal face of $\Gamma(f|_{\mathbb{R}^J})$ (maximal with respect to the inclusion of faces) where the linear function takes this value, i.e.,

$$d \ := \ d(q, \Gamma(f|_{\mathbb{R}^J})) \quad \text{and} \quad \Delta \ := \ \Delta(q, \Gamma(f|_{\mathbb{R}^J}))$$

(here we put $q_j := 0$ for $j \notin J$.) Then $\Delta \in \Gamma_\infty(f)$ because $\min_{i\in J} q_j < 0$. Furthermore, we have asymptotically as $t \to 0^+$,

$$f(\phi(t)) = f_\Delta(x^0) t^d + \text{ higher-order terms},$$

where $x^0 := (x_1^0, \ldots, x_n^0)$ with $x_j^0 := 1$ for $j \notin J$ (note that the polynomial f_Δ does not depend on x_j for $j \notin J$). Combining this with Condition (b) gives $f_\Delta(x^0) = 0$, which contradicts the assumption (ii). $\square$

3.2 The stability of compactness of an algebraic set.

In the rest of this chapter we study the stable compactness of real algebraic sets, which is easier to check than compactness.

Definition 3.6. The set $\mathcal{Z}(f)$ is called *stably compact* if there is $\epsilon > 0$ such that $\mathcal{Z}(f+g)$ is compact for all polynomials $g\colon \mathbb{R}^n \to \mathbb{R}$ with $\Gamma(g) \subseteq \Gamma(f)$ and $|g| < \epsilon$.

By definition, the set $\mathcal{Z}(f)$ is stably compact if, and only if, remains compact for all sufficiently small perturbations of the "Newton" coefficients of the polynomial f.

Lemma 3.7. *The following conditions are equivalent:*

(i) $f_\Delta \neq 0$ *on* $(\mathbb{R} \setminus \{0\})^n$ *for all* $\Delta \in \Gamma_\infty(f)$.

(ii) One of the following statements holds

- *(ii1)* $f_\Delta > 0$ *on* $(\mathbb{R} \setminus \{0\})^n$ *for all* $\Delta \in \Gamma_\infty(f)$.
- *(ii2)* $f_\Delta < 0$ *on* $(\mathbb{R} \setminus \{0\})^n$ *for all* $\Delta \in \Gamma_\infty(f)$.

Proof. It suffices to show the implication (i) $\Rightarrow$ (ii). Assume this is not the case, which means that there exist faces $\Delta_1, \Delta_2 \in \Gamma_\infty(f)$ such that $f_{\Delta_1} > 0 > f_{\Delta_2}$ on $(\mathbb{R} \setminus \{0\})^n$. We may assume further that these faces are adjacent, i.e., $\Delta := \Delta_1 \cap \Delta_2 \neq \emptyset$. Then $\Delta \in \Gamma_\infty(f)$. By assumption, $f_\Delta \neq 0$ on $(\mathbb{R} \setminus \{0\})^n$. Fix $x^0 := (x_1^0, \ldots, x_n^0) \in (\mathbb{R} \setminus \{0\})^n$, and without loss of generality, we may assume that $f_\Delta(x^0) > 0$. By definition, there exists a vector q with $\min_{j=1,\ldots,n} q_j < 0$ such that $\Delta = \Delta(q, \Gamma(f))$. A simple calculation shows that

$$f_{\Delta_2}(t^{q_1} x_1^0, \ldots, t^{q_n} x_n^0) \;=\; t^d f_\Delta(x^0) + \text{ higher-order terms,}$$

where $d := d(q, \Gamma(f))$. Since $f_\Delta(x^0) > 0$, this implies that $f_{\Delta_2}(t^{q_1}x_1^0, \ldots, t^{q_n}x_n^0) > 0$ for all $t > 0$ small enough, which contradicts the fact that $f_{\Delta_2} < 0$ on $(\mathbb{R} \setminus \{0\})^n$. $\square$

In what follows, let $\mathcal{P}(x) := \sum_{\alpha \in \Gamma(f) \cap \mathbb{Z}_+^n} |x^\alpha|$ and for each face Δ of the polyhedron $\Gamma(f)$, set $\mathcal{P}_\Delta(x) := \sum_{\alpha \in \Delta \cap \mathbb{Z}_+^n} |x^\alpha|$. By definition, the functions $\mathcal{P}$ and $\mathcal{P}_\Delta$ are positive on $(\mathbb{R} \setminus \{0\})^n$.

Remark 3.8. Let $\widetilde{\mathcal{P}}(x) := \sum_\alpha |x^\alpha|$, where the sum is taken over all the vertices of $\Gamma(f)$. Then there exist positive constants c_1, c_2, and R such that

$$c_1\mathcal{P}(x) \ \leq \ \widetilde{\mathcal{P}}(x) \ \leq \ c_2\mathcal{P}(x) \quad \text{for all} \quad x \in \mathbb{R}^n.$$

Indeed, the right-hand inequality clearly holds with $c_2 := 1$. To see the left-hand inequality, let $v^1, \ldots, v^s$ be the vertices of the polyhedron $\Gamma(f)$. Then, for each $\alpha \in \Gamma(f)$, there exist non-negative real numbers $\lambda_1, \ldots, \lambda_s$, with $\lambda_1 + \cdots + \lambda_s = 1$, such that

$$\alpha = \lambda_1 v^1 + \cdots + \lambda_s v^s.$$

Consequently, for all $x \in \mathbb{R}^n$ we have

$$\begin{aligned} |x^\alpha| = |x^{\lambda_1 v^1 + \cdots + \lambda_s v^s}| \ &= \ (|x^{v^1}|)^{\lambda_1} \cdots (|x^{v^s}|)^{\lambda_s} \\ &\leq \ \lambda_1 |x^{v^1}| + \cdots + \lambda_s |x^{v^s}| \\ &\leq \ |x^{v^1}| + \cdots + |x^{v^k}| \ = \ \widetilde{\mathcal{P}}(x). \end{aligned}$$

Hence $\frac{1}{\#(\Gamma(f) \cap \mathbb{Z}_+^n)}\mathcal{P}(x) \leq \widetilde{\mathcal{P}}(x)$, which completes the proof.

The following lemma is a version at infinity of (Bui & Pham, 2016, Theorem 3.2). In the lemma, the equivalent of the statements (i) and (ii) was proved in (Gindikin, 1974; Mikhailov, 1967); for the sake of completeness we give a proof, which is different from the ones in these papers.

Lemma 3.9. *The following conditions are equivalent*

(i) $f_\Delta > 0$ *on* $(\mathbb{R} \setminus \{0\})^n$ *for all* $\Delta \in \Gamma_\infty(f)$.

(ii) There exist positive constants c_1, c_2, and R such that

$$c_1 \mathcal{P}(x) \ \leq \ f(x) \ \leq \ c_2 \mathcal{P}(x) \quad \text{for all} \quad \|x\| > R. \tag{3.1}$$

(iii) f is Newton non-degenerate at infinity and there exists $R > 0$ such that $f(x) \geq 0$ for all $\|x\| > R$.

Proof. (i) $\Rightarrow$ (ii) Suppose that f is written as $f = \sum_\alpha a_\alpha x^\alpha$. We have for all $x \in \mathbb{R}^n$,

$$f(x) \ \leq \ \sum_\alpha |a_\alpha||x^\alpha| \ \leq \ \max_\alpha |a_\alpha| \sum_\alpha |x^\alpha| \ \leq \ \max_\alpha |a_\alpha| \mathcal{P}(x),$$

and so the right-hand inequality in (3.1) holds with $c_2 := \max_\alpha |a_\alpha| > 0$.

Suppose the left-hand inequality in (3.1) was false. By the Curve Selection Lemma at infinity (see Theorem 1.14), then we could find analytic curves $\phi\colon (0, \epsilon) \to \mathbb{R}^n, t \mapsto (\phi_1(t), \ldots, \phi_n(t))$, and $c\colon (0, \epsilon) \to \mathbb{R}$ such that the following assertions hold:

(a) $\|\phi(t)\| \to +\infty$ as $t \to 0^+$;

(b) $c(t) > 0$ for $t \in (0, \epsilon)$, $c(t) \to 0$ as $t \to 0^+$;

(c) $c(t)\mathcal{P}(\phi(t)) > f(\phi(t))$ for $t \in (0, \epsilon)$.

Let $J := \{j \mid \phi_j \not\equiv 0\} \subset \{1, \ldots, n\}$. By Condition (a), $J \neq \emptyset$. We can expand the functions $c(t)$ and $\phi_j(t)$ for $j \in J$, in terms of the parameter, say

$$\begin{aligned} c(t) &= c_0 t^p + \text{ higher-order terms} \\ \phi_j(t) &= x_j^0 t^{q_j} + \text{ higher-order terms,} \end{aligned}$$

where $c_0 \neq 0, x_j^0 \neq 0$ and $p, q_j \in \mathbb{Q}$. By conditions (a) and (b), $c_0 > 0$ and $p > 0 > \min_{j \in J} q_j$.

If $\mathbb{R}^J \cap \Gamma(f) = \emptyset$, then for each $\alpha \in \Gamma(f)$, there exists an index $j \notin J$ such that $\alpha_j > 0$. Consequently,

$$\mathcal{P}(\phi(t)) \ \equiv \ \sum_{\alpha \in \Gamma(f) \cap \mathbb{Z}_+^n} |\phi(t)^\alpha| \ \equiv \ \sum_{\alpha \in \Gamma(f) \cap \mathbb{Z}_+^n} \left(\prod_{j \in J} |\phi_j(t)^{\alpha_j}| \prod_{j \notin J} |\phi_j(t)^{\alpha_j}| \right) \ \equiv \ 0.$$

Similarly, we also have $f(\phi(t)) \equiv 0$, which contradicts Condition (c).

Therefore, $\mathbb{R}^J \cap \Gamma(f) \neq \emptyset$. Let d be the minimal value of the linear function $\sum_{j\in J} \alpha_j q_j$ on $\mathbb{R}^J \cap \Gamma(f)$ and let Δ be the maximal face of $\Gamma(f)$ where this linear function takes its minimum value. Then $\Delta \in \Gamma_\infty(f)$ since $\min_{j\in J} q_j < 0$. Furthermore, we have asymptotically as $t \to 0^+$,

$$\begin{aligned} c(t)\mathcal{P}(\phi(t)) &= c_0 \mathcal{P}_\Delta(x^0) t^{d+p} + \text{ higher-order terms}, \\ f(\phi(t)) &= f_\Delta(x^0) t^d + \text{ higher-order terms}, \end{aligned}$$

where $x^0 := (x_1^0, \ldots, x_n^0)$ with $x_j^0 := 1$ for $j \notin J$. Note that $\mathcal{P}_\Delta(x^0) > 0$ and $f_\Delta(x^0) > 0$. Therefore, by Condition (c), we get

$$d + p \leq d,$$

which contradicts the fact that $p > 0$.

(ii) $\Rightarrow$ (iii) The left-hand inequality in (3.1) shows that $f(x) \geq 0$ for all $\|x\| > R$.

Take any $x^0 \in (\mathbb{R} \setminus \{0\})^n$ and $\Delta \in \Gamma_\infty(f)$. By definition, there exists a vector $q \in \mathbb{R}^n$ with $\min_{j=1,\ldots,n} q_j < 0$ such that $\Delta = \Delta(q, \Gamma(f))$. Consider the monomial curve

$$\phi\colon (0, +\infty) \to \mathbb{R}^n, \quad t \mapsto (x_1^0 t^{q_1}, \ldots, x_n^0 t^{q_n}).$$

Clearly, $\|\phi(t)\| \to +\infty$ as $t \to 0^+$. Furthermore, we have asymptotically as $t \to 0^+$,

$$\begin{aligned} \mathcal{P}(\phi(t)) &= \mathcal{P}_\Delta(x^0) t^d + \text{ higher-order terms}, \\ f(\phi(t)) &= f_\Delta(x^0) t^d + \text{ higher-order terms}, \end{aligned}$$

where $d := d(q, \Gamma(f))$. Since $\mathcal{P}_\Delta(x^0) > 0$, it follows from (3.1) that $f_\Delta(x^0) > 0$. In particular, f is Newton non-degenerate at infinity.

(iii) $\Rightarrow$ (i) Take any $\Delta \in \Gamma_\infty(f)$. We first show that $f_\Delta \geq 0$ on $(\mathbb{R} \setminus \{0\})^n$. On the contrary, suppose that $f_\Delta(x^0) < 0$ for some $x^0 \in (\mathbb{R} \setminus \{0\})^n$. By definition, there exists a vector $q \in \mathbb{R}^n$ with $\min_{j=1,\ldots,n} q_j < 0$ such that $\Delta = \Delta(q, \Gamma(f))$. Consider the monomial curve

$$\phi\colon (0, +\infty) \to \mathbb{R}^n, \quad t \mapsto (x_1^0 t^{q_1}, \ldots, x_n^0 t^{q_n}).$$

Clearly, $\|\phi(t)\| \to +\infty$ as $t \to 0^+$. Furthermore, we have asymptotically as $t \to 0^+$,

$$f(\phi(t)) \;=\; f_\Delta(x^0)t^d + \text{ higher-order terms},$$

where $d := d(q, \Gamma(f))$. Since $f_\Delta(x^0) < 0$, it follows that $f < 0$ on the curve ϕ, which contradicts our assumption.

Therefore, $f_\Delta \geq 0$ on $(\mathbb{R} \setminus \{0\})^n$, and by continuity, we have $f_\Delta \geq 0$ on $\mathbb{R}^n$.

We next show that $f_\Delta > 0$ on $(\mathbb{R} \setminus \{0\})^n$. By contradiction, suppose that $f_\Delta(x^0) = 0$ for some $x^0 \in (\mathbb{R} \setminus \{0\})^n$. Since $f_\Delta \geq 0$ on $\mathbb{R}^n$, it follows that x^0 is a global minimizer of f_Δ on $\mathbb{R}^n$, and so x^0 is a critical point of f_Δ. Therefore,

$$f_\Delta(x^0) = \frac{\partial f_\Delta(x^0)}{\partial x_1} = \cdots = \frac{\partial f_\Delta(x^0)}{\partial x_n} = 0,$$

which contradicts the non-degeneracy condition of f. □

The following result presents necessary and sufficient conditions for the stable compactness in terms of the Newton polyhedron of the defining polynomial.

Theorem 3.10 (Compare Theorem 0.6)**.** *The following conditions are equivalent:*

(i) $\mathcal{Z}(f)$ is stably compact.

(ii) $f|_{\mathbb{R}^J} \not\equiv 0$ for all $J \subset \{1, \ldots, n\}$ and $f_\Delta \neq 0$ on $(\mathbb{R} \setminus \{0\})^n$ for all $\Delta \in \Gamma_\infty(f)$.

(iii) $f|_{\mathbb{R}^J} \not\equiv 0$ for all $J \subset \{1, \ldots, n\}$ and one of the following statements holds

(iii1) $f_\Delta > 0$ on $(\mathbb{R} \setminus \{0\})^n$ for all $\Delta \in \Gamma_\infty(f)$.

(iii2) $f_\Delta < 0$ on $(\mathbb{R} \setminus \{0\})^n$ for all $\Delta \in \Gamma_\infty(f)$.

(iv) $f|_{\mathbb{R}^J} \not\equiv 0$ for all $J \subset \{1, \ldots, n\}$ and there exist $\sigma \in \{-1, 1\}$ and constants $c_1 > 0, c_2 > 0$, and $R > 0$ such that

$$c_1\mathcal{P}(x) \;\leq\; \sigma f(x) \;\leq\; c_2\mathcal{P}(x) \quad \textit{for all} \quad \|x\| > R. \tag{3.2}$$

(v) $f|_{\mathbb{R}^J} \not\equiv 0$ *for all* $J \subset \{1, \ldots, n\}$, f *is Newton non-degenerate at infinity, and there exist* $\sigma \in \{-1, 1\}$ *and* $R > 0$ *such that* $\sigma f(x) \geq 0$ *for all* $\|x\| > R$.

Proof. The equivalences (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) follow immediately from Lemmas 3.7 and 3.9. Hence, it suffices to show (i) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (i).

(i) $\Rightarrow$ (iii) By assumption, the set $\mathcal{Z}(f)$ is compact. Thanks to Theorem 3.3, $f|_{\mathbb{R}^J} \not\equiv 0$ for all $J \subset \{1, \ldots, n\}$. Replacing f by $-f$ if necessary, we may assume that f is bounded from below and $f_\Delta \geq 0$ on $(\mathbb{R} \setminus \{0\})^n$ for all $\Delta \in \Gamma_\infty(f)$. We will show that (iii1) holds. On the contrary, suppose that there exist $x^0 \in (\mathbb{R} \setminus \{0\})^n$ and $\Delta \in \Gamma_\infty(f)$ such that $f_\Delta(x^0) = 0$. This implies that Δ contains at least two vertices, say Δ_1 and Δ_2. Note that all the coordinates of the vertices Δ_1 and Δ_2 are even integer numbers because f is bounded from below. This implies easily that $f_{\Delta_1}(x^0) > 0$ and $f_{\Delta_2}(x^0) > 0$.

Now, for each $\epsilon > 0$ consider the polynomial $g_\epsilon(x) := -\epsilon x^{\Delta_1}$. Clearly, $g_\epsilon(x) = -\epsilon f_{\Delta_1}(x)$, $\Gamma(g_\epsilon) \subset \Gamma(f)$, and $\Gamma_\infty(f + g_\epsilon) = \Gamma_\infty(f)$ for all $\epsilon > 0$ small enough. Furthermore, we have

$$\begin{aligned}(f + g_\epsilon)_\Delta(x^0) &= f_\Delta(x^0) + g_{\epsilon,\Delta}(x^0) = -\epsilon f_{\Delta_1}(x^0) < 0,\\ (f + g_\epsilon)_{\Delta_2}(x^0) &= f_{\Delta_2}(x^0) > 0.\end{aligned}$$

By Theorem 3.3, $\mathcal{Z}(f + g_\epsilon)$ is not compact, a contradiction.

(iv) $\Rightarrow$ (i) Without loss of generality, we may assume that (iv) holds with $\sigma = 1$. Let f be written as $f = \sum_\alpha a_\alpha x^\alpha$ and set

$$\epsilon := \min\left\{\frac{c_1}{2}, \min_\alpha |a_\alpha|\right\} > 0,$$

where the second minimum is taken over all the vertices of $\Gamma(f)$.

Take any polynomial $g\colon \mathbb{R}^n \to \mathbb{R}$ with $\Gamma(g) \subseteq \Gamma(f)$ and $|g| < \epsilon$. By definition, $\Gamma_\infty(f + g) = \Gamma_\infty(f)$. Furthermore, for all $x \in \mathbb{R}^n$ we have

$$|g(x)| \leq |g| \sum_{\alpha \in \Gamma(f)} |x^\alpha| \leq \frac{c_1}{2}\mathcal{P}(x).$$

It follows from (3.2) that

$$\frac{c_1}{2}\mathcal{P}(x) \;\leq\; (f+g)(x) \;\leq\; \left(\frac{c_1}{2}+c_2\right)\mathcal{P}(x) \quad \text{for} \quad \|x\| > R. \tag{3.3}$$

Consequently, for all $J \subset \{1, \ldots, n\}$ we have $(f+g)|_{\mathbb{R}^J} \not\equiv 0$ since otherwise $\mathcal{P}|_{\mathbb{R}^J} \equiv 0$, and hence $f|_{\mathbb{R}^J} \equiv 0$ by (3.2), a contradiction.

Furthermore, from (3.3) and Lemma 3.9, we deduce that $(f+g)_\Delta > 0$ on $(\mathbb{R} \setminus \{0\})^n$ for all $\Delta \in \Gamma_\infty(f+g)$.

Therefore, in view of Theorem 3.5, the set $\mathcal{Z}(f+g)$ is compact. $\square$

Let us make some final comments.

Remark 3.11. (i) The criteria presented in this paper can be easily extended to examine the (stable) compactness of basic closed semi-algebraic sets. To see this, let X be a basic closed semi-algebraic set defined by

$$X := \{x \in \mathbb{R}^n \mid g_1(x) = 0, \ldots, g_l(x) = 0,\ h_1(x) \geq 0, \ldots, h_m(x) \geq 0\},$$

where $g_1, \ldots, g_l, h_1, \ldots, h_m$ are polynomial functions on $\mathbb{R}^n$. It is easy to see that X is compact if, and only if, the set

$$Y := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid g_1(x) = 0, \ldots, g_l(x) = 0,\ h_1(x) - y_1^2 = 0, \ldots, h_m(x) - y_m^2 = 0\},$$

is compact. Then the statement follows because Y is the zero set of the polynomial function

$$\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, (x, y) \mapsto [g_1(x)]^2 + \cdots + [g_l(x)]^2 + [h_1(x) - y_1^2]^2 + \cdots + [h_m(x) - y_m^2]^2.$$

(ii) In the two-dimensional case, the criteria remain valid if one checks only the codimension one faces. Unfortunately, this is not true in the general case as can be seen with the polynomial $f(x, y, z) := x^2 + (y - z)^2$.

(iii) Finally, we would like to mention that the established criteria can be checked, at least in principle. Indeed, given a polynomial function $f\colon \mathbb{R}^n \to \mathbb{R}$, there are algorithms with polynomial time complexity that can generate all faces of the Newton polyhedron $\Gamma(f)$ (see (Fukuda, 2004; Fukuda & Rostal, 1994; K. Fukuda & Margot, 1997; Murty & Chung, 1995)). Moreover, for each face $\Delta \in \Gamma_\infty(f)$, it is not hard to see that the problem of checking positivity (or non-negativity) of the polynomial f_Δ (corresponding to the face Δ) on $(\mathbb{R} \setminus \{0\})^n$ can be reduced to the problem of minimizing polynomial functions over semi-algebraic sets; on the other hand, for each polynomial optimization problem, by using standard results about the existence of sums of squares certificates (i.e., Positivstellensätze), we can construct an appropriate sequence of computationally feasible semidefinite programming relaxations (Lasserre's hierarchy), whose optimal values converge monotonically, increasing to the optimal value of the original problem. For more detailed information on the subject, see the surveys (Laurentl, 2009; Scheiderer, 2009) and the monographs (Hà & Phạm, 2017; Lasserrel, 2009; Marshall, 2008), as well as references therein.

In summary, this chapter establishes the following main results:

- Giving a necessary condition and a sufficient condition for the compactness of an algebraic set $\mathcal{Z}(f)$ which is defined by a real polynomial function which is bounded either from above or from below.
- The necessary and sufficient criteria for the stable compactness of $\mathcal{Z}(f)$.

Conclusions

The main goals of this thesis are to study properties of a class of functions satisfying non-degenerate conditions. Singularity Theory and Semi-algebraic Geometry are main tools for our study. Our main results include:

- Investigating the global monodromy of a family polynomial $\{f_t\}$ restricting on an algebraic set in which the Newton polyhedrons of $\{f_t\}$ are independent from t and satisfy the non-degenerated condition. (see Theorem 2.7).

- Giving a necessary condition and a sufficient condition for the compactness of an algebraic set $\mathcal{Z}(f)$ which is defined by a real polynomial function which is bounded either from above or from below. This implies the necessary and sufficient criteria for the stable compactness of $\mathcal{Z}(f)$. (see Theorem 3.3, Theorem 3.5 and Theorem 3.10).

List of Author's Related Papers

[BP-1] P. P. Phạm and T. S. Phạm, *Compactness criteria for real algebraic set and Newton polyhedra,* Forum Mathematicum, **30** (6)(2018).

[BP-2] T. T. Nguyen, P. P. Phạm and T. S. Phạm, *Bifurcation Sets and Global Monodromies of Newton Non-degenerate Polynomials on Algebraic Sets,* PRIMS Kyoto Univ., **55** (4) (2019).

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